Operator equations, multiscale concepts and complexity

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Abstract

In this paper, we review several recent developments centering upon the application of multiscale basis methods for the numerical solution of operator equations with special emphasis on complexity questions. In particular, issues like preconditioning, matrix compression, construction of special wavelet bases and adapted error estimators are addressed.

1 Introduction

An obvious byproduct of the enormous increase of computing power is the likewise tremendeous increase of greed for handling even more complex tasks. As a result numerical analysis has developed a number of new exciting facets. One of them is the intertwinement of linear algebra tasks with analytical concepts and complexity.

The objective of this article is to highlight some of the recent developments in this regard. The material presented here covers essentially the contents of the lecture given by the first author as well as the topics addressed in the subsequent workshop on "Multiscale Techniques" held on August 4, 1995, by all authors. Due to the survey nature of the article the material covers joint work with others as well and the close collaboration with S. Dahlke, R. Hochmuth, and S. Prössdorf deserves to be credited explicitely.

1.1 A General Model Problem

A classical and central task in numerical simulation is the solution of systems of (eventually) linear equations. When these systems stem from discretizing some continuous problem their size N may become very large involving several millions of unknowns. In this range the amount of storage and the number of floating point operations needed to solve such a problem are reasonable complexity measures. In fact, computability essentially requires that both quantities remain proportional to the problem size N, i.e., the number of unknowns. Of course, this can generally not be achieved for arbitrary linear systems unless the system matrices are 'essentially' diagonal. The point is that for a wide class of problems the matrices are essentially diagonal in a sense to be made more precise below. In fact, at this point some analytical background of the problem comes into play. In order to bring out the main ideas we will consider therefore the following (admittedly oversimplified) model class of problems. Given two Hilbert spaces H_1, H_2, let

$$A : H_1 \to H_2$$

be a linear operator which is boundedly invertible, i.e.,

$$\|Au\|_{H_2} \sim \|u\|_{H_1}, \quad u \in H_1. \tag{1.1}$$

Here H_1, H_2 typically stand for Sobolev spaces on various types of domains or manifolds and $a \sim b$ will always mean that a and b can be bounded by constant multiples of each other where the constants are independent of any parameters a and b may depend on. Given $f \in H_2$, find $u \in H_1$ such that

$$Au = f. \tag{1.2}$$

In the course of the discussion we will add more information about A.

1.2 Discretization

To keep matters as simple as possible we will focus on Galerkin schemes for solving (1.2) approximately. Denoting by $\langle \cdot, \cdot \rangle$ the inner product for some Hilbert space H, $H_1 \subset H \subset H_2$, we fix a finite dimensional space $S_h \subset H_1$ and seek for $u_h \in S_h$ such that

$$\langle Au_h, v_h \rangle = \langle f, v_h \rangle, \quad v_h \in S_h. \tag{1.3}$$

Here we assume for simplicity that $H_2 = H_1^*$, the dual of H_1 relative to the dual form $\langle \cdot, \cdot \rangle$ on $H_1 \times H_2$. Fixing a basis for S_h then leads to a linear system of equations

$$\mathbf{A}_h \mathbf{u}_h = \mathbf{f}_h. \tag{1.4}$$

When one tries to minimize the complexity of solving (1.2) approximately one faces essentially two issues:

- Solve (1.4) in (almost) *linear time.*

By this we mean that when ε_h is the discretization error achieved by the Galerkin scheme and $N_h = \dim S_h$ then the amount of storage and the number of floating point operations needed to solve (1.4) with accuracy $\mathcal{O}(\varepsilon_h)$ is $\mathcal{O}(N_h)$ for $N_h \to \infty$ (or $\mathcal{O}(N_h(\log N_h)^a)$, $N_h \to \infty$). Such solvers will be referred to as *asymptotically optimal* (a.o.).

Note that asymptotic optimality refers only to the *discrete* problem. A completely different but likewise important question is:

- Find (nearly) optimal discretizations.

The probably most practicable approach to this problem is to choose the trial spaces S_h *adaptively.* This already suggests working with *sequences* of trial spaces $S_{h_j} = S_j$, $j \in \mathbb{N}_0$, where S_{j+1} is a refinement of S_j, i.e., $S_j \subset S_{j+1}$. Thus, a key step is to study a whole sequence $\mathcal{S} = \{S_j\}_{j \in \mathbb{N}_0}$ of trial spaces and exploit the interaction of different scales of discretization to extract as much as possible (asymptotic) information about the object one is trying to recover. This by itself is a familiar concept represented e.g. by multigrid methods [H]. Here we will focus on a variant which has been strongly influenced and partly initiated by recent developments in the theory and applications of wavelets. It will be shown that such a suitably extended basis orientated approach allows one to handle both of the above problems in a rather unified fashion for a wide range of problems.

The paper is organized as follows. In Section 2 we present several types of operator equations and identify the principal difficulties obstructing the realization of a.o. schemes. This covers scalar elliptic partial differential equations, saddle point problems arising e.g. from Stokes' equations as well as boundary integral equations.

In Section 3 we describe a general multiresolution framework for sequences of trial spaces and associated multiscale bases. The main issues in this context are inverse and approximation properties resulting in norm equivalences. Furthermore, we sketch some principles of constructing multiscale bases with these properties.

Section 4 is devoted to applying these concepts to the problem types introduced in Section 2. In particular, we will indicate in which way the analysis of the respective problem class specifies particular demands on the corresponding tools in terms of appropriate multiscale bases. A brief discussion of the realization of such demands and corresponding construction principles is contained in Section 5. This concerns e.g. the construction of pairs of trial spaces for pressure and velocity on bounded domains in $\mathbb{R}^n$ which satisfy the Ladyšenskaja-Babuška-Brezzi condition as well as Riesz bases defined on closed manifolds satisfying certain regularity and moment conditions.

Finally, in Section 6 we address adaptive strategies based on *reliable* and *efficient* a-posteriori local error estimates. In principle, these concepts apply to elliptic problems both for differential and integral operators.

2 More about the Scope of Problems

In this section we describe several types of model problems and their intrinsic features hampering an efficient numerical solution.

(I) Scalar elliptic problems

Suppose that $a(\cdot,\cdot) : H^\tau \times H^\tau \to \mathbb{R}$ is a bilinear symmetric form such that

$$a(\cdot,\cdot) \sim \|\cdot\|^2_{H^\tau}\,. \tag{2.1}$$

Here H^τ stands for a Sobolev space over some bounded domain $\Omega \subset \mathbb{R}^n$, e.g. $H^\tau(\Omega)$ or $H^\tau_0(\Omega)$. Given $f \in (H^\tau)^*$ (the adjoint of H^τ), find $u \in H^\tau$ such that

$$a(u,v) = (f,v)_\Omega, \quad v \in H^\tau, \tag{2.2}$$

where $(u,v)_\Omega = \int\limits_\Omega u(x)v(x)\,\mathrm{d}x$. Here the operator $A : H^\tau \to (H^\tau)^*$ is defined by $(Au,v)_\Omega = a(u,v)$. Typical examples are $A = -\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $A = -\operatorname{div}(\mathbf{A}\operatorname{grad})$ where $\mathbf{A} = \mathbf{A}(x)$ is a positive definite matrix, and $H_1 = H^1_0(\Omega)$, $H_2 = H^{-1}(\Omega)$, or $A := -\Delta + \lambda I$, $\lambda > 0$, on Ω and $H_1 = H^1(\Omega)$, $H_2 = H^1(\Omega)^*$.

When S_h is a trial space spanned by compactly supported functions such that the diameter of their supports is $\mathcal{O}(h)$ the matrix $\mathbf{A}_h$ in (1.4) is symmetric positive definite and sparse, i.e., when refining the meshsize the number of nonzero entries in each row remains uniformly bounded. Thus the storage needed for the matrix and right hand side is $\mathcal{O}(N_h)$. However, when the spatial domain Ω has dimension larger than one a direct method for solving (1.4) would cause expanding the storage by an additional positive power of N_h due to *fill in*. On the other hand, the speed of convergence of an iterative method for symmetric positive definite problems depends on the spectral condition number of the matrix. A necessary condition for an a.o. scheme is that any fixed error reduction requires a finite number of iterations independent of the problem size N_h. Unfortunately, in the present situation it is well known that

$$\operatorname{cond}(\mathbf{A}_h) = \|A_h\| \left\|A_h^{-1}\right\| \sim h^{-2\tau}. \tag{2.3}$$

Thus the central objective in this case is to *precondition* (1.4), i.e., to find a positive definite matrix $\mathbf{C}_h$ such that the application of $\mathbf{C}_h\mathbf{c}$, $\mathbf{c} \in I\!R^{N_h}$, requires only $\mathcal{O}(N_h)$ operations and

$$\mathrm{cond}(\mathbf{C}_h\mathbf{A}_h) = \mathrm{cond}(\mathbf{C}_h^{1/2}\mathbf{A}_h\mathbf{C}_h^{1/2}) \sim 1. \tag{2.4}$$

(II) Saddle point problems

Consider the following simplified model for an instationary viscous incompressible fluid flow

$$\begin{aligned} \tfrac{\partial u}{\partial t} - \nu\Delta u + \mathrm{grad}\, p &= f && \text{in } \Omega \times (0,T) \\ \mathrm{div}\, u &= 0 && \text{in } \Omega \times (0,T) \\ u(\cdot,0) &= u_0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \times (0,T) \\ \int_\Omega p(x,t)\,\mathrm{d}x &= 0 && \text{for } t \in (0,T) \end{aligned} \tag{2.5}$$

known as Stokes equation which may be viewed as the linearized version of the Navier-Stokes equations for small velocities $u \in H_0^1(\Omega)^n$, $n = 2, 3$. Here p stands for the pressure. Defining

$$a(u,v) = \sum_{i=1}^{n} (\mathrm{grad}\, u_i, \mathrm{grad}\, v_i)_\Omega, \quad b(u,q) = (\mathrm{div}\, u, q)_\Omega,$$

the weak formulation of (2.5) for the stationary case reads: find $(u,p) \in H_0^1(\Omega)^n \times (L_2(\Omega)/I\!R)$ such that

$$\begin{aligned} \nu a(u,v) + b(v,p) &= (f,v)_\Omega, && v \in H_0^1(\Omega)^n, \\ b(u,q) &= 0, && q \in L_2(\Omega)/I\!R. \end{aligned} \tag{2.6}$$

Choosing finite dimensional trial spaces $V_h \subset H_0^1(\Omega)^n$, $M_h \subset L_2(\Omega)/I\!R$, (2.6) gives rise to the linear system of equations

$$\begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h^T \\ \mathbf{B}_h & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ 0 \end{pmatrix}. \tag{2.7}$$

Here $\mathbf{A}_h$ is positive definite and symmetric and corresponds to the matrices arising in (I). The whole system (2.7) is indefinite though and it is well-known that the stable solvability of (2.7) is equivalent to the validity of the Ladyšenskaja-Babuška-Brezzi condition (LBB) [BF, GR]

$$\inf_{q \in M_h} \sup_{v \in V_h} \frac{b(v,q)}{\|v\|_{H_0^1(\Omega)^n}\, \|q\|_{L_2(\Omega)/I\!R}} \geq \beta > 0, \quad h \to 0. \tag{2.8}$$

This condition seriously constrains the choice of trial spaces. For $n = 3$ the list of known admissible pairs of finite element trial spaces is significantly shorter than that for $n = 2$. Thus the first issue is the construction of *admissible discretizations*. Next, given admissible pairs of trial spaces the indefinite system (2.7) has to be solved efficiently. Several approaches have been proposed in the literature, e.g. [BP, BWY]. Roughly speaking, (2.7) can be efficiently solved iteratively whenever good preconditioners for $\mathbf{A}_h$ and for the *Schur complement* $\mathbf{K}_h := \mathbf{B}_h\mathbf{A}_h^{-1}\mathbf{B}_h^T$ are available. As mentioned above any good preconditioner from (I) works for $\mathbf{A}_h$ here as well. Moreover, in the stationary case $\mathbf{K}_h$ is a

boundedly invertible operator on $L_2(\Omega)/\mathbb{R}$ which is therefore well conditioned. However, when using a fully implicit scheme for the time dependent case, the condition number of $\mathbf{K}_h$ deteriorates when $\nu\Delta t$ becomes small. It will be seen that both issues constructing admissible discretizations and preconditioning can be handled in a unified fashion by a multiscale basis approach.

Similar saddle point problems arise through mixed formulations of the second order problems from (I) where, in particular, the same form $b(u,q) = (\operatorname{div} u, q)_\Omega$ appears motivating the study of *divergence free* basis functions [DKU1, U].

(III) Boundary integral equations

The computation of electrostatic fields or scattering from obstacles are typical problems where a boundary value problem is to be solved on the *exterior* of some bounded domain. For example, consider

$$\Delta U = 0 \quad \text{on} \quad \mathbb{R}^3 \setminus \Omega, \qquad U = f \quad \text{on} \quad \Gamma = \partial\Omega. \tag{2.9}$$

There are several ways of transforming (2.9) into an integral equation over Γ. For instance, the so-called *indirect method* yields the equivalent operator equation

$$Au = f$$

where $A = I + 2K$ and when Ω is a polyhedral domain

$$(Ku)(x) = \left(\frac{1}{2} - \theta_\Omega(x)\right) u(x) + \frac{1}{4\pi} \int_\Gamma \frac{n_y^T(x-y)}{|x-y|^3} u(y)\, \mathrm{d}s_y. \tag{2.10}$$

Here $\theta_\Omega(x)$ denotes the interior angle between adjacent facets when x lies on an edge and n_y is the exterior normal of the surface at $y \in \Gamma$.

Alternatively, the *direct method* yields

$$Au = Vu = \left(\frac{1}{2} I + K\right) f$$

where

$$(Vu)(x) = \int_\Gamma \frac{u(y)}{4\pi\,|x-y|}\, \mathrm{d}s_y. \tag{2.11}$$

K and V are called *double* and *single layer potential operators*, respectively. Thus in both cases we observe the following principal advantage:

- A problem defined on an unbounded 3-dimensional domain is reduced to a problem on a 2-dimensional bounded domain.

An obvious disadvantage is:

- Discretization of (2.10) and (2.11) based on collocation or Galerkin schemes gives rise to *densely populated* matrices.

Both formulations have different specific advantages and disadvantages. To explain this we need the notion of Sobolev spaces $H^s(\Gamma)$ defined on Γ. Depending on the regularity of the surface they are for a certain range of exponents naturally defined as trace spaces (see [A, G] for details). For smooth boundaries Γ, K is known to be a pseudodifferential operator of order $r = 0$. The order r may be defined as the degree of homogeneity of the *symbol* of the operator. Here it means that $L_2(\Gamma)$ is mapped boundedly invertible into itself by K.

By contrast V is known to have order $r = -1$. The single layer potential formulation is better suited for handling different types of boundary conditions. For coupling boundary element and finite element techniques, the fact $r \neq 0$ has a similar effect as in the case of differential operators: the matrices $\mathbf{A}_h$ get increasingly ill conditioned when $\dim S_h$ grows.

In summary, one faces two types of obstructions:

- The complexity of conventional methods for such problems is at least $\mathcal{O}((\dim S_h)^2)$ due to the densely populated matrices.
- When dealing with operators of order $r \neq 0$ the efficient solution of the systems of equation is in addition hampered by growing condition numbers.

Our goal is to describe some general concepts providing a unified platform for dealing with the above mentioned obstructions, namely preconditioning, construction of good discretizations (e.g. pairs of trial spaces satisfying (LBB) in (II) or wavelets on Γ in (III)), sparsification of densely populated matrices, analysing stability and convergence, and adaptive construction of trial spaces for a wide range of cases.

3 Multiscale Bases and Wavelets

Wavelets offer a very promising potential for a number of applications such as signal/image analysis and compression or data smoothing as well as for the treatment of operator equations. At least that is true when dealing with specific domains like $\mathbb{R}^n$ or the torus. This may change dramatically when dealing with more complex domain geometries which usually exclude the application of wavelets per se in comparison with other modern methods. Nevertheless, one can extract a number of promising features which may still work well when put in a suitably general framework. To identify relevant properties which are not restricted to the convenient affinely invariant setting on all of $\mathbb{R}^n$ or to the periodized case, we will proceed now working first with a general framework of multiresolution in some Hilbert space H which may represent e.g. $L_2(\mathbb{R}^n)$, $L_2(\Gamma)$, $H_0^s(\Omega)$, $H^s(\Gamma)$ etc.

3.1 Multiresolution in H

$\mathcal{S}$ will always denote a sequence of nested closed subspaces S_j of H whose union is dense in H, i.e.,

$$S_0 \subset S_1 \subset \ldots \subset H, \quad \operatorname{clos}_H \left(\bigcup_{j=0}^{\infty} S_j \right) = H. \tag{3.1}$$

The spaces S_j will have the form

$$S_j = S(\Phi_j) := \mathrm{clos}_H\left(\mathrm{span}(\Phi_j)\right), \quad \Phi_j = \{\varphi_{j,k} : k \in \Delta_j\}, \tag{3.2}$$

where Δ_j is an (at most countable) index set, and the Φ_j are stable bases in the sense that (uniformly in j)

$$\|\mathbf{c}\|_{\ell_2(\Delta_j)} \sim \left\|\Phi_j^T\mathbf{c}\right\|_H. \tag{3.3}$$

Here $\|\mathbf{c}\|_{\ell_2(\Delta_j)} := \left(\sum_{k\in\Delta_j}|c_k|^2\right)^{1/2}$ and

$$\Phi_j^T\mathbf{c} := \sum_{k\in\Delta_j} c_k\varphi_{j,k} \tag{3.4}$$

is a convenient short hand notation treating the basis Φ_j formally as a vector whose components are the basis functions. Due to the assumed stability the order of summation in (3.4) when Δ_j is infinite does not matter and every element in $S(\Phi_j)$ has the form (3.4) for a unique $\mathbf{c} \in \ell_2(\Delta_j)$.

In the spirit of wavelets updating a current approximation $v_j \in S_j$ can be facilitated by identifying a "suitable" complement $W_j = S(\Psi_j)$ of $S(\Phi_j)$ in $S(\Phi_{j+1})$,

$$S(\Phi_{j+1}) = S(\Phi_j) \oplus S(\Psi_j), \tag{3.5}$$

where $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$ may be viewed as "wavelets". Of course the bases Ψ_j should be uniformly stable as well by which we mean that

$$\left\|\Psi_j^T\mathbf{d}\right\|_H \sim \|\mathbf{d}\|_{\ell_2(\nabla_j)} \tag{3.6}$$

holds uniformly in j. Clearly, there is a continuum of possible complements and it has to be made precise what "suitable" means in this context.

3.2 Multiscale Transformations

A first hint in this regard can be obtained as follows. Thinking of the $\varphi_{j,k}$, for instance, as the piecewise linear Courant hat functions relative to some triangulation, it is clear that the coefficients c_j in (3.4) reflect "geometric" information on displacements while the coefficients $\mathbf{d}$ in $\Psi_j^T\mathbf{d}$ of the j-th update have the character of *differences*. Thus, writing

$$S(\Phi_j) = S(\Phi_0)\bigoplus_{\ell=0}^{j-1} S(\Psi_\ell),$$

any $v_j \in S(\Phi_j)$ can be written in *single scale representation* as

$$v_j = \Phi_j^T\mathbf{c} \tag{3.7}$$

or in *multiscale form* as

$$v_j = \sum_{\ell=-1}^{j-1} \Psi_\ell^T\mathbf{d}_\ell \tag{3.8}$$

where we set for convenience $\Psi_{-1} := \Phi_0$. Trying to exploit the advantages of both representations brings in the transformation

$$\mathbf{T}_j : (\mathbf{d}_{-1}, \ldots, \mathbf{d}_{j-1}) \mapsto \mathbf{c} \tag{3.9}$$

that takes one set of expansion coefficients into the other one. For this to be of any use the $\mathbf{T}_j$ should be

- *efficient*, i.e., the application of $\mathbf{T}_j$ should require only the order of $\#\Delta_j$ operations,
- *well conditioned*, i.e.,
$$\|\mathbf{T}_j\| \left\|\mathbf{T}_j^{-1}\right\| = \mathcal{O}(1), \quad j \in I\!N_0. \tag{3.10}$$

Efficiency essentially requires all basis functions in Φ_j and Ψ_j to have small compact support, a point to be addressed later on again in connection with various specifications. (3.10) entails some deeper analytic consequences to be addressed next.

3.3 Riesz Bases and Biorthogonality

One can show [D1, D2] that (3.10) holds if and only if the multiscale basis

$$\Psi = \bigcup_{j=-1}^{\infty} \Psi_j \tag{3.11}$$

is a Riesz basis of H and if there exists another *biorthogonal* Riesz basis $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : (j,k) \in \nabla\}$, $\nabla = \{(j,k) : k \in \nabla_j,\ j = -1, 0, 1, 2, \ldots\}$, i.e.,

$$\left\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \right\rangle = \delta_{(j,k),(j',k')}, \quad (j,k),(j',k') \in \nabla, \tag{3.12}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in H, and every $v \in H$ has a *unique* expansion

$$v = \sum_{(j,k)\in\nabla} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} = \sum_{(j,k)\in\nabla} \langle v, \psi_{j,k} \rangle \, \tilde{\psi}_{j,k} \tag{3.13}$$

such that

$$\|v\|_H^2 \sim \sum_{(j,k)\in\nabla} \left| \left\langle v, \tilde{\psi}_{j,k} \right\rangle \right|^2 \sim \sum_{(j,k)\in\nabla} |\langle v, \psi_{j,k} \rangle|^2 . \tag{3.14}$$

Of course, when Ψ is an orthonormal basis one has $\Psi = \tilde{\Psi}$ and the transformations $\mathbf{T}_j$ are orthogonal so that (3.10) is trivially valid. But orthonormal bases are usually hard to construct. Orthonormality tends to interfere with locality and we will encounter examples below where orthogonal complements are *not* the best choice. So it is important to make use of the flexibility offered by biorthogonality.

However, while biorthogonality appears to be necessary the question arises whether it is also sufficient for (3.14). It turns out that, in general, this is not the case. Nevertheless, it is important to note that additional properties needed to ensure (3.14) are not properties

of the complements and their bases but rather properties of the multiresolution sequence $\mathcal{S}$ *and* of the *dual* sequence $\tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j\in\mathbb{N}_0}$ where

$$\tilde{S}_j = S\left(\bigcup_{\ell=-1}^{j-1} \tilde{\Psi}_\ell\right) \tag{3.15}$$

which can be phrased as approximation and regularity properties. To describe this, suppose $\omega(\cdot,t) : H \to H$, $t > 0$, is a uniformly bounded family of subadditive functionals such that $\lim_{t\to 0}\omega(v,t) = 0$, $v \in H$. We call such ω a *modulus*. Moreover, note that the pair of Riesz bases induces canonical truncation operators

$$Q_j v := \sum_{\ell=-1}^{j-1}\sum_{k\in\nabla_\ell} \left\langle v, \tilde{\psi}_{\ell,k}\right\rangle \psi_{\ell,k}$$

which, by (3.12) and (3.14) are uniformly bounded projectors with ranges S_j satisfying

$$Q_\ell Q_j = Q_\ell, \quad \ell \le j. \tag{3.16}$$

Finally, note that uniform stability of the complement bases Ψ_j means that

$$\|(Q_{j+1} - Q_j)v\|_H^2 \sim \sum_{k\in\nabla_j} \left|\left\langle v, \tilde{\psi}_{j,k}\right\rangle\right|^2 \tag{3.17}$$

so that (3.14) is equivalent to

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_H^2\right)^{1/2}, \quad v \in H, \tag{3.18}$$

where $Q_{-1} := 0$.

Theorem 3.1 [D2] *Suppose $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{N}_0}$ is a sequence of uniformly H-bounded projectors with ranges $\mathcal{S} = \{S_j\}_{j\in\mathbb{N}_0}$ satisfying* (3.16). *Let $\tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j\in\mathbb{N}_0}$ be the ranges of the adjoints $\mathcal{Q}^* = \{Q_j^*\}_{j\in\mathbb{N}_0}$. If there exists a modulus $\omega(\cdot,t)$ on H such that*

$$\inf_{v_j\in V_j} \|v - v_j\|_H \lesssim \omega(v, 2^{-j}), \quad v \in H, \tag{3.19}$$

and

$$\omega(v_j, t) \lesssim \left(\min\{1, t2^j\}\right)^\gamma \|v_j\|_H, \quad v_j \in V_j, \tag{3.20}$$

for some $\gamma > 0$ and $V_j = S_j$, $V_j = \tilde{S}_j$, then (3.18) *holds.*

Here $a \lesssim b$ (equivalent to $b \gtrsim a$) means that a can be bounded by a constant multiple of b independent of parameters a, b may depend on.

A few points are worth mentioning. Typical proofs of the Riesz basis property make essential use of Fourier techniques and are thus restricted to shift-invariant multiresolution on the whole Euclidian space or on the torus. Criteria of the above type aim at replacing

Fourier techniques, thereby offering tools for dealing with bounded domains or more general manifolds.

Secondly, in practice one usually has direct access only to the trial spaces $S_j = S(\Phi_j)$ while a suitable biorthogonal basis Ψ is yet to be constructed. The above formulation suggests a strategy to do that. As a first step, given the $\{\Phi_j\}$, construct biorthogonal collections $\tilde{\Phi}_j$ for each j such that the corresponding projectors

$$Q_j v = \sum_{k \in \Delta_j} \langle v, \tilde{\varphi}_{j,k} \rangle \, \varphi_{j,k} \tag{3.21}$$

are uniformly bounded and satisfy (3.16) which is easily seen to be equivalent to the $\tilde{\Phi}_j$ being refinable (see below for details). In a second step one has to identify for each j stable bases Ψ_j for the complements

$$W_j := (Q_{j+1} - Q_j) S_{j+1}. \tag{3.22}$$

We will comment later on ways to that as well.

Before doing that we specialize the above results to the characterization of Sobolev spaces.

3.4 Norm Equivalences for Sobolev Spaces

Let us denote by $H^s, s \in \mathbb{R}$, a scale of Sobolev spaces either on a bounded domain or on a sufficiently smooth manifold, where for $s < 0$ H^s is to be understood as $(H^{-s})^*$.

Specializing the arguments employed in the proof of Theorem 3.1 yields the following result [D2].

Theorem 3.2 *Suppose that*

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2} \lesssim 2^{-sj} \|v\|_{H^s}, \quad v \in H^s, \tag{3.23}$$

for $s \leq d$ when $V_j = S_j$ and $s \leq \tilde{d}$ when $V_j = \tilde{S}_j$. Moreover, assume that

$$\|v_j\|_{H^s} \lesssim 2^{sj} \|v_j\|_{L_2}, \quad v_j \in V_j, \tag{3.24}$$

for $0 < s \leq \gamma \leq d$ when $V_j = S_j$ and $0 < s \leq \tilde{\gamma} \leq \tilde{d}$ when $V_j = \tilde{S}_j$. Then

$$\|v\|_{H^s} \sim \left(\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \right)^{1/2} \tag{3.25}$$

holds for $s \in (-\tilde{\gamma}, \gamma)$.

Furthermore, one can show the validity of the one sided norm estimates [Sch]

$$\begin{aligned} \|v\|_{H^s} &\lesssim \left(\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \right)^{1/2}, \ s \in (-\tilde{d}, \gamma), \\ \|v\|_{H^s} &\gtrsim \left(\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \right)^{1/2}, \ s \in (-\tilde{\gamma}, d). \end{aligned} \tag{3.26}$$

One way to read Theorem 3.2 is that the mapping

$$\Lambda_s v := \sum_{j=0}^{\infty} 2^{sj}(Q_j - Q_{j-1})v \tag{3.27}$$

acts like a Bessel potential operator as a shift in the scale H^s, i.e.,

$$\|\Lambda_s v\|_{H^t} \sim \|v\|_{H^{s+t}}, \quad s+t \in (-\tilde{\gamma}, \gamma). \tag{3.28}$$

Similar constructive characterizations can be derived for a wide class of Besov-type spaces [D3]. Such characterizations may be viewed as an attempt to replace Fourier techniques by a more flexible tool like multiresolution to facilitate a unified treatment of various cases with possibly different geometrical background.

Relation (3.28) is a key to the following observations.

4 Preconditioning

4.1 A General Fact

Let H^s be as above. The common ground for the problems (I), (II), (III) may be extracted as follows. Suppose that $A : H^s \to H^{s-r}$ is boundedly invertible, i.e.,

$$\|Av\|_{H^{s-r}} \sim \|v\|_{H^s}, \quad v \in H^s. \tag{4.1}$$

Moreover, suppose that $\mathcal{S}$, $\tilde{\mathcal{S}}$, $\mathcal{Q}$ are as above. Note that the Galerkin conditions (1.3) are equivalent to determining $u_j \in S_j$ such that

$$Q_j^* A u_j = Q_j^* f. \tag{4.2}$$

The scheme is said to be $(s, s-r)$-stable if

$$\left\|Q_j^* A v_j\right\|_{H^{s-r}} \sim \|v_j\|_{H^s}, \quad v_j \in S_j. \tag{4.3}$$

Note that when A is selfadjoint, $s = \tau$, $r = 2\tau$, and (4.3) is equivalent to (2.1).

To see how (4.3) relates to the class of problems described in (III) it is convenient to employ the framework of pseudo-differential operators. Classical pseudo-differential operators of class $\Psi^{(r)}(\mathbb{R}^n)$ on $\mathbb{R}^n$ have the form

$$(Au)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i \xi\cdot(x-y)}\sigma(x,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad u \in C_0^\infty(\mathbb{R}^n),$$

where the symbol $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to the class $S^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies, in particular,

$$\left|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)\right| \le c_{\alpha,\beta}(1+|\xi|)^{r-|\alpha|}, \quad x, \xi \in \mathbb{R}^n,\ \alpha, \beta \in \mathbb{Z}_+^n.$$

Corresponding classes $\Psi^{(r)}(\Gamma)$ of operators on other domains or manifolds can be defined by requiring that their transports relative to an atlas and partition of unity belong to $\Psi^{(r)}(\mathbb{R}^n)$. It is known that if the following conditions (**A1**) hold,

- A is strongly elliptic, i.e., the Gårding inequality

$$\mathrm{Re}\,\sigma_0(x,\xi) \gtrsim |\xi|^r\,, \quad \xi \in I\!R^n,$$

holds uniformly for the principal parts σ_0 of the symbols of all transports of A,

- A is injective,

then (4.1) is valid for all $s \in I\!R$ (see e.g. [DPS]). It is known that for smooth surfaces Γ in (III) the single and double layer potential operators are pseudo-differential operators in $\Psi^{(r)}(\Gamma)$, $r = -1, 0$, respectively.

It follows from [HW] that under the assumptions (**A1**) the Galerkin scheme is $(\tau, -\tau)$-stable where we always use $2\tau = r$. Moreover, one can then show under the previous assumptions on $\mathcal{S}$ that the scheme is $(s, s-2\tau)$-stable for $2\tau - d \le s \le \tau$ and that the deviation of the solution u_j of (4.2) from the solution u of (1.2) can be estimated by

$$\|u - u_j\|_{H^t} \lesssim 2^{j(t-s)}\,\|u\|_{H^s}\,, \tag{4.4}$$

where $-d + 2\tau \le t < \gamma$, $t \le s$, $\tau \le s \le d$, giving $\mathcal{O}(2^{-j(2d-\tau)})$ as optimal convergence rate.

To solve (4.2) we will assume in the following that

$$\tilde{\gamma} > -\frac{r}{2}, \quad \gamma > \frac{r}{2}. \tag{4.5}$$

Denoting by Ψ^j the multiscale bases $\cup_{\ell=-1}^{j-1}\Psi_\ell$ of S_j, let

$$\mathbf{A}_{\Psi^j} = \left(\langle A\psi_{j',k'}, \psi_{j,k}\rangle\right)_{(j,k),(j',k')\in\nabla^j}\,, \quad \nabla^j = \bigcup_{\ell=-1}^{j-1}\left(\{j\}\times\nabla_j\right)$$

be the stiffness matrix of A relative to Ψ^j.

Theorem 4.1 [DPS] *Let $\mathbf{D}_s$ be the diagonal matrix with entries*

$$2^{s\ell}\delta_{(\ell,k),(\ell',k')}, \quad (\ell,k),(\ell',k') \in \nabla^j.$$

Then

$$\mathrm{cond}\left(\mathbf{D}_{-r/2}\mathbf{A}_{\Psi^j}\mathbf{D}_{-r/2}\right) = \mathcal{O}(1), \quad j \to \infty, \tag{4.6}$$

where $\mathrm{cond}(B) = \|B\|\,\|B^{-1}\|$ *and* $\|\cdot\|$ *is the spectral norm.*

The proof is an immediate application of the above norm equivalences. In fact, putting $w_j := \Lambda_{r/2}v_j$ one has by (4.5), (4.3) and (3.28)

$$\begin{aligned} \|w_j\|_{H^0} &\sim \left\|\Lambda_{r/2}v_j\right\|_{H^0} \sim \|v_j\|_{H^{r/2}} \\ &\sim \left\|Q_j^*Av_j\right\|_{H^{-r/2}} \sim \left\|\Lambda^*_{-r/2}Q_j^*AQ_j\Lambda_{-r/2}v_j\right\|_{H^0} \end{aligned} \tag{4.7}$$

where we have used that $\Lambda_s^{-1} = \Lambda_{-s}$. It is not hard to verify that the matrix representation of $B_j := \Lambda^*_{-r/2}Q_j^*AQ_j\Lambda_{-r/2}$ relative to the basis Ψ^j is $\mathbf{D}_{-r/2}\mathbf{A}_{\Psi^j}\mathbf{D}_{-r/2}$. Since (4.7) says that the B_j are uniformly bounded and boundedly invertible operators on H^0 the assertion follows.

A few comments on the above results are in order:

- Neither need the operators $B_j = \Lambda^*_{-r/2} Q_j^* A Q_j \Lambda_{-r/2}$ be symmetric nor is the order of A to be nonnegative.

- When A is a partial differential operator and the bases Φ_j consist of compactly supported functions the matrices $\mathbf{A}_{\Phi_j}$ would be sparse while the $\mathbf{A}_{\Psi^j}$ would generally be more densely populated because of the basis functions from low levels. In this case the original goal of designing a.o. schemes can not be met when computing and storing $\mathbf{A}_{\Psi^j}$. Noting that
$$\mathbf{A}_{\Psi^j} = \mathbf{T}_j^* \mathbf{A}_{\Phi_j} \mathbf{T}_j \tag{4.8}$$
where $\mathbf{T}_j$ is the multiscale transformation from (3.9) one only has to store $\mathbf{A}_{\Phi_j}$. Consequently, one can apply $\mathbf{A}_{\Psi^j}$ by successively applying $\mathbf{T}_j$, $\mathbf{A}_{\Phi_j}$ and $\mathbf{T}_j^*$ which is a process of order $\dim S_j$ occupying only storage of that same order. If A has positive order and is self-adjoint it should be noted that closely related alternative preconditioners referred to as BPX preconditioners or multilevel Schwarz schemes [BPX, DK, O1, O2, GO] are available. These techniques do not require explicit knowledge of the complement bases Ψ_j.

  Moreover, when dealing with adaptively refined finite element spaces, it can be shown that although direct estimates of the form (3.19) or (3.23) are no longer available, the BPX scheme still gives rise to uniformly bounded condition numbers [DK].

  Remark 4.1 *In this case, either of these techniques as well as any multigrid scheme is asymptotically optimal, which can be seen as follows. Choosing an initial level j_0, let ε_{j_0} denote the discretization error realized by the Galerkin approximation u_{j_0} relative to the energy norm. By Theorem* 4.1*, a fixed finite number of conjugate gradient iterations suffices to determine the solution of* (4.2) *within a tolerance of ε_{j_0}. Since $S_{j_0} \subset S_{j_0+1}$ one can take that solution as a starting point for solving* (4.2) *in S_{j_0+1}. $\varepsilon_{j_0+1}/\varepsilon_{j_0}$ is a constant independent of j_0 and again a finite number of conjugate gradient steps are needed to reduce the error to ε_{j_0+1}. By the above remarks each conjugate gradient step takes $\mathcal{O}(\dim S_j)$ operations and $\dim S_j$ typically grows exponentially so that the whole computational work remains proportional to that required by the finite number of steps on the finest level which, in turn, remains proportional to $\dim S_j$ independently of j.*

- Even for certain nonsymmetric operators, e.g. when the symmetry is destroyed by a lower order term, schemes like GMRES exhibit the same performance.

- When A has a global Schwartz kernel the situation changes since $\mathbf{A}_{\Phi_j}$ is densely populated and each iteration requires the order of $(\dim S_j)^2$ operations while direct solvers are even more expensive. However, $\mathbf{A}_{\Psi^j}$ turns out to be much closer to a sparse matrix as explained in the following section.

4.2 Matrix Compression

For univariate periodic problems involving a zero order operator it was observed in [BCR] that the entries of $\mathbf{A}_{\Psi^j}$ exhibit a certain decay. This turns out to remain true under much more general circumstances.

To describe the mechanism one has to be a bit more specific about the setting. An important class of applications involving operators with global kernels are boundary integral equations as described in example (III). In this case the domain Γ is the boundary of some domain $\Omega \subset I\!R^3$ which will be assumed here to be parametrically represented as a union of closed smooth patches

$$\Gamma = \bigcup_{i=1}^{N} \Gamma_i \tag{4.9}$$

where $\mathrm{meas}(\Gamma_i \cap \Gamma_\ell) = 0$, $i \neq \ell$, and any two adjacent patches join at least continuously so that Γ is a Lipschitz boundary. Thus, for each i there exists a smooth mapping $\kappa_i : \square \to \Gamma_i$, where $\square = [0,1]^2$ is the reference domain (see [DS1] for more details). Practical schemes realizing surfaces of this form for essentially arbitrary topology can be found in [HM, Re].

The crucial condition on the multiscale basis functions $\psi_{j,k}$ defined on Γ can be formulated now as follows. Denote by Π_{d^*} the space of bivariate polynomials of order at most d^* (i.e., degree $d^* - 1$). Ψ is said to have patch-wise vanishing moments of order d^* if

$$\int_{\square} P(x)(\psi_{j,k} \circ \kappa_i)(x)\,\mathrm{d}x = 0, \quad P \in \Pi_{d^*},\ i = 1, \ldots, N. \tag{4.10}$$

Of course, analogous conditions can be formulated for higher dimensional domains or manifolds. We have chosen the special version (4.10) here for later reference.

The relevance of conditions of type (4.10) becomes clear when noting that for $A \in \Psi^{(r)}(\Gamma)$ and any atlas $\{\tilde{\Gamma}_m, \tilde{\kappa}_m\}_{m=1}^{\tilde{N}}$ with partition of unity $\{\xi_m \in C_0^\infty(\tilde{\Gamma}_m),\ m = 1, \ldots, \tilde{N}\}$ the Schwartz kernels $K_{m,m'}$ of the transported operators

$$A_{m,m'} = \tilde{\kappa}_m^{-1} \circ \xi_m A \xi_m \circ \tilde{\kappa}_{m'}$$

satisfy

$$\left|\partial_x^\alpha \partial_y^\beta K_{m,m'}(x,y)\right| \leq c_{\alpha,\beta}\,|x-y|^{-(2+r+|\alpha|+|\beta|)}, \quad x, y \in I\!R^2, \quad x \neq y. \tag{4.11}$$

Therefore, if $\mathrm{dist}\,(\Omega_{j,k}, \Omega_{j',k'}) > 0$,

$$\begin{aligned}
\langle A\psi_{j',k'}, \psi_{j,k}\rangle_{\Gamma_i} &= \int_{\Gamma_i} (A\psi_{j',k'})(x)\psi_{j,k}(x)\,\mathrm{d}s_x \\
&= \int_{\square} (A\psi_{j',k'})(\kappa_i(y))\psi_{j,k}(\kappa_i(y))\kappa_i'(y)\,\mathrm{d}y \\
&= \int_{\square}\int_{\square} \sum_{\Omega_{j',k'} \cap \Gamma_\ell \neq \emptyset} K(\kappa_i(y), \kappa_\ell(z))\psi_{j',k'}(\kappa_\ell(z))\psi_{j,k}(\kappa_i(y))\kappa_i'(y)\kappa_\ell'(z)\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

where $\Omega_{j,k} := \operatorname{supp} \psi_{j,k}$.

Defining

$$g(y,z) := K(\kappa_i(y), \kappa_\ell(z))\kappa_i'(y)\kappa_\ell'(z)$$

and writing

$$g(y,z) = P(y,z) + R(y,z), \quad R := g - P,$$

where for each z $P(y,z)$ is the Taylor polynomial of $g(\cdot,z)$ in y of order d^* around y_0, (4.10) says that

$$\int_\square \left(\int_\square g(y,z) \left(\psi_{j,k} \circ \kappa_i\right)(y)\, \mathrm{d}y \right) \left(\psi_{j',k'} \circ \kappa_\ell\right)(z)\, \mathrm{d}z$$
$$= \int_\square \int_\square R(y,z) \left(\psi_{j,k} \circ \kappa_i\right)(y) \left(\psi_{j',k'} \circ \kappa_\ell\right)(z)\, \mathrm{d}y\, \mathrm{d}z.$$

Likewise expanding $R(y,\cdot)$ around $z_0 \in \Omega_{j',k'}$, using (4.10) relative to z and taking (4.11) into account yields for the entries of the stiffness matrix $\mathbf{A}_{\Psi^J}$ the estimate

$$|\langle A\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim \frac{2^{-(j+j')(1+d^*)}}{\left(\operatorname{dist}\left(\Omega_{j,k}, \Omega_{j',k'}\right)\right)^{2+r+2d^*}} \tag{4.12}$$

[DPS, PS, Sch] provided that $\operatorname{dist}\left(\Omega_{j,k}, \Omega_{j',k'}\right) \geq \max\{c2^{-j}, c2^{-j'}\}$. It is clear that the order d^* of moment conditions determines the decay of these entries. Here and below J will denote the finest discretization level.

To avoid logarithmic terms in the complexity analysis, we need an even more subtle estimate. Indeed, the above analysis requires regularity of the kernel on $(\Omega_{j,k} \cap \Gamma_i) \times (\Omega_{j',k'} \cap \Gamma_l)$. It is not hard to see that there are $\mathcal{O}(J2^{2J})$ matrix coefficients which cannot be estimated by the above procedure. To deal with these entries let us denote by

$$\Omega^S_{(j',k')} := \operatorname{sing\,supp} \psi_{j',k'}$$

the *singular support* of $\psi_{j',k'}$. For the wavelets introduced in the subsequent sections, the singular support consists of the boundaries of the sub-domains $\widehat{\mathcal{T}}^{j'} = \kappa_l(\mathcal{T}^{j'}) \subseteq \Omega_{j,k} \cap \Gamma_i$. Here $\mathcal{T}^j$ is supposed to be the maximal domain such that $\psi_{j,k} \circ \kappa_l = p_j \in \Pi_D$ is a polynomial of degree D on $\mathcal{T}^j \subset \square$.

If $j' < j$ and $\operatorname{dist}\left(\Omega_{j,k}, \Omega_{j',k'}\right) \lesssim 2^{-j'}$, the estimate

$$|\langle A\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim \frac{2^{-j(1+d^*)}2^{j'}}{\left(\operatorname{dist}\left(\Omega_{j,k}, \Omega^S_{j',k'}\right)\right)^{r+d^*}} \tag{4.13}$$

holds [Sch].

Based on these estimates the subsequent analysis proceeds in the following steps:

- A level dependent a-priori truncation rule can be designed in such a way that upon replacing all those entries staying below the corresponding thresholds, the resulting compressed matrix $\mathbf{A}^c_{\Psi^J}$ is sparse containing only $\mathcal{O}(\dim S_J)$ non-vanishing entries.

- This can be used to derive consistency estimates of the form

$$\|(A_J - A_J^c)\, u\|_{H^{s-r}} \lesssim a^{-r-2d^*}\, 2^{J(s-t)}\, \|u\|_{H^t} \tag{4.14}$$

  where $a < 1$ is fixed, A_J, A_J^c are the finite dimensional operators corresponding to $\mathbf{A}_{\Psi^J}$ and $\mathbf{A}^c_{\Psi^J}$, respectively, and the range of the parameters s and t is $-d + r \leq s < \gamma$, $-\gamma < t \leq d$, d and γ taken from (3.26).

- The preconditioned matrix $\mathbf{B}_J = \mathbf{D}_{-r/2}\mathbf{A}_{\Psi^J}\mathbf{D}_{-r/2}$ is by Theorem 4.1 uniformly boundedly invertible on $L_2(\Gamma)$. Thus, a Schur-lemma argument leads to estimates for the remainder $\mathbf{A}_{\Psi^J} - \mathbf{A}^c_{\Psi^J}$ with respect to the spectral norm.

- A perturbation argument combined with (4.14) then gives stability and convergence results which can be summarized as follows.

Suppose that

$$d < d^* + r.$$

We define the compressed matrix $\mathbf{A}^c_{\Psi^J}$ in two steps. Firstly, let, for $(j,k), (j',k') \in \nabla^J$, $a_{(j,k),(j',k')} := \langle A\psi_{j',k'}, \psi_{j,k}\rangle$ and

$$a^1_{(j,k),(j',k')} := \begin{cases} a_{(j,k),(j',k')}, & \text{if } \operatorname{dist}(\Omega_{(j,k)}, \Omega_{(j',k')}) \leq \mathcal{B}_{j,j'}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

Here the parameter $\mathcal{B}_{j,j'}$ is chosen such that for some $d' \in (d, d^* + r)$

$$\mathcal{B}_{j,j'} \sim \max\, \{a\, 2^{-j}, a\, 2^{-j'}, a 2^{(J(2d'-r) - j'(d^*+d') - j(d^*+d'))/(2d^*+r)}\}. \tag{4.16}$$

In a second step, we set

$$(\mathbf{A}^c_{\Psi^J})_{(j,k),(j',k')} := \begin{cases} a^1_{(j,k),(j',k')}, & j' \leq j \text{ and } \operatorname{dist}(\Omega_{(j,k)}, \Omega^S_{(j',k')}) \leq \mathcal{B}^S_{j,j'}, \\ & j \leq j' \text{ and } \operatorname{dist}(\Omega^S_{(j,k)}, \Omega_{(j',k')}) \leq \mathcal{B}^S_{j,j'}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}$$

where the truncation parameters $\mathcal{B}^S_{j,j'}$ controlling the distance from the singular support are given by

$$\mathcal{B}^S_{j,j'} \sim \max\, \{a'\, 2^{-j}, a'\, 2^{-j'}, a' 2^{(J(2d'-r) - \max\{j,j'\}d^* - (j+j')d')/(d^*+r)}\}. \tag{4.18}$$

The parameters a, a' are fixed constants. For instance, a determines the bandwidth in the block matrices $\mathbf{A}^c_{J,J} = (a^c_{(J,k),(J,k')})_{k,k' \in \nabla_J}$. a, a' have to be chosen sufficiently large such that the stability of the preconditioned compressed matrix is preserved [DPS, Sch]. We stress that a, a' are independent of J.

Theorem 4.2 *Let $\mathcal{S}$, $\mathcal{Q}$ satisfy the assumptions in Theorem* 3.2 *such that* (4.5) *holds. Moreover, assume that Ψ satisfies the patch-wise moment conditions* (4.10) *of order d^* where*

$$d^* > d - r. \tag{4.19}$$

Then the above compression strategy gives rise to matrices $\mathbf{A}^c_{\Psi^J}$ containing only $\mathcal{O}(\dim S_J)$ non-vanishing entries. Suppose that the assumptions of Section 4.1 *hold so that the scheme*

(4.2) is $(s, s-r)$-stable, $2\tau = r$, for $2\tau - d \le s \le \tau$. Then the compressed system possesses a unique solution u_J^c realizing asymptotically optimal accuracy

$$\|u - u_J^c\|_{H^t} \lesssim 2^{J(t-s)} \|u\|_{H^s} \tag{4.20}$$

where $-d + r \le t < \gamma$, $t \le s$, $\frac{r}{2} \le s \le d$ and u is the exact solution of $Au = f$. Moreover, the matrices $\mathbf{B}_J^c = \mathbf{D}_{-r/2} \mathbf{A}_{\Psi^J}^c \mathbf{D}_{-r/2}$ have uniformly bounded condition numbers.

Note that the Galerkin scheme gives optimal convergence rates $\mathcal{O}(2^{-(2d-r)J})$ relative to low norms.

We summarize the required conditions on the wavelet basis. To realize an asymptotically optimal balance between accuracy and efficiency, the regularity γ of Ψ, the regularity $\tilde{\gamma}$ of the dual basis $\tilde{\Psi}$, the order of vanishing moments d^* and the order of exactness d of the trial spaces S_J have to be related in the following way:

regularity	$\gamma > \frac{r}{2}$ conformity	$\tilde{\gamma} > -\frac{r}{2}$ preconditioning
order	d	convergence rate $2^{-J(2d+2-r)}$
vanishing moments	$d^* > (\geq) d - r$	

Note also that, on account of (4.19), optimal results can therefore not be guaranteed for *orthogonal* wavelets when $r \le 0$. One rather has to construct then $\mathcal{S}$ and $\mathcal{Q}$ such that the dual multiresolution $\tilde{\mathcal{S}}$ has higher order of exactness, a point to be addressed later again.

The above theorem says that even if the convergence behavior of an iterative solver is governed by the condition numbers we still end up with an asymptotically optimal scheme as described in the previous subsection, provided that the compressed matrices $\mathbf{A}_{\Psi^J}^c$ can be computed with accuracy comparable to the discretization error on the highest level at computational costs which remain proportional to $\dim S_J$. To indicate that this actually can be done is the objective of the following section which summarizes some recent results from [DS2].

4.3 Adaptive Quadrature

Up to this point we have assumed that the matrix entries $\langle A\psi_{j,k}, \psi_{j',k'} \rangle$ are given exactly. Of course, in general they have no closed analytical representation.

However, in principle, one can accurately compute the stiffness matrix $\mathbf{A}_{\Phi_J}$ relative to the single scale basis Φ_J (see Section 5.4). Now the *multiscale transformation* $\mathbf{T}_J$ from (3.9) yields

$$\mathbf{A}_{\Psi^J} = \mathbf{T}_J^* \mathbf{A}_{\Phi_J} \mathbf{T}_J$$

which is the system matrix with respect to the multiscale basis. Note that the original matrix $\mathbf{A}_{\Phi_J}$ has $\mathcal{O}(2^{4J})$ nonzero coefficients. Thus, the overall complexity is still $\mathcal{O}(2^{4J}) = \mathcal{O}((\dim S_J)^2)$ which rules out this way of computing the nonzero coefficients of the compressed matrix $\mathbf{A}_J^c$.

To find a more economic strategy, let us first recall that we have already derived an a priori criterion to decide whether a matrix coefficient must be computed or can be neglected.

In addition, observe that dist $(\Omega_{j,k}, \Omega_{j',k'}) > \mathcal{B}_{j,j'}$ implies that dist $(\Omega_{j_1,k_1}, \Omega_{j_1',k_1'}) > \mathcal{B}_{j_1,j_1'}$ holds for $\Omega_{j_1,k_1} \subset \Omega_{j,k}$ and $\Omega_{j_1',k_1} \subset \Omega_{j',k'}$, $j_1 \geq j, j_1' \geq j'$. Thus, we do not have to check condition (4.15) or (4.17) for all pairs $(j,k),(j',k')$. Exploiting the hierarchical structure of multiscale bases we need at most $\mathcal{O}(2^{2J})$ checks to decide whether an entry has to be computed or not.

An accurate computation of the remaining nonzero coefficients is a difficult task. The significant coefficients have to be computed approximately by numerical integration. In order to approximate the matrix coefficients $\langle A\psi_{j,k}, \psi_{j',k'}\rangle$, we have to evaluate integrals of the form

$$\int_{\widehat{\mathcal{T}}^j} \int_{\widehat{\mathcal{T}}^{j'}} K(\hat{x},\hat{y})\psi_{j,k}(\hat{x})\psi_{j',k'}(\hat{y})ds_{\hat{x}}ds_{\hat{y}} \tag{4.21}$$

over curved quadrilaterals $\widehat{\mathcal{T}}_x^j = \kappa_i(\mathcal{T}_x^j)$ and $\widehat{\mathcal{T}}_y^{j'}$ where $\hat{x} = \kappa_i(x)$, $\hat{y} = \kappa_l(y)$.

In terms of the present parametrizations (4.21) takes the form

$$\int_{\mathcal{T}^j} \int_{\mathcal{T}^{j'}} p(x,y)p_j(x)p_{j'}(y)dx\,dy, \tag{4.22}$$

where $p_j, p_{j'}$ are polynomials of degree $\hat{d} \geq d-1$ satisfying

$$\|p_j\|_{W^{s,\infty}(\mathcal{T}^j)} \lesssim 2^{(s+1)j}. \tag{4.23}$$

Since numerical integration returns only an approximate value of the integral, it causes an additional error in the solution u_J^{CQ} of the fully discrete system. Thus, the design of a fully discrete method requires carefully monitoring the overall accuracy while still preserving efficiency.

The problem of quadrature has to be seen in close connection with compression and the special features of multiscale bases. Basis functions from coarser scales introduce large domains of integration while requiring high accuracy. In particular, on the coarsest scale $j, j' = -1$ the full accuracy $2^{-J(2d'-r)}$ depending on J is required while on the highest scale $j, j' = J$ fixed accuracy suffices. In fact, since $|\langle A\psi_{j,k}, \psi_{j,k'}\rangle| \lesssim 2^{jr}$, this accuracy is actually independent of J. Thus, many entries have to be computed only with low accuracy while high accuracy is only required for a small part of the matrix. Using the analysis of matrix compression as a guideline, a careful balancing of the various effects shows that most matrix entries $\langle A\psi_{j,k}, \psi_{j',k'}\rangle$ must be computed with a precision

$$e_{(j,k),(j',k')} \lesssim 2^{-J(2d'-r)}2^{\max\{j,j'\}(d'+1)}2^{\min\{j,j'\}(d'+1)}2^{-2\max\{j,j'\}}$$

with some $d' > d$.

Our fully discretizedGalerkin method is based on product-type Gaussian formulas for approximating inner and outer integrals

$$\int_{\tau_x} \int_{\tau_y} p_x(x)p_y(y)dx\,dy = Q_x^D \otimes Q_y^D\,(p_x \cdot p_y), \quad \text{for all} \quad p_x, p_y \in \Pi_D, \tag{4.24}$$

where the domains τ_x and τ_y are congruent to $\square$. According to the previous remarks, the error estimate for the quadrature method has much in common with estimating matrix coefficients relative to wavelet bases.

Lemma 4.1 *Let $Q^D_{\tau_x} \otimes Q^D_{\tau_y}$ be a product-type Gaussian quadrature method of order D and $\tau_x \subset \mathcal{T}^j$, $\tau_y \subset \mathcal{T}^{j'}$. Furthermore, suppose that A is a boundary integral operator of order r over a piecewise analytic boundary surface Γ. Let in local parametrization the kernel be denoted by $p(x,y)$ and $G(x,y) := p(x,y)p_j(x)p_{j'}(y)$. If $\tau_x \cap \tau_y = \emptyset$, then there exists a constant c such that the estimate*

$$\begin{aligned} &\Big|\int_{\tau_x}\int_{\tau_y} G(x,y)dxdy - Q^D_{\tau_x} \otimes Q^D_{\tau_y}(G)\Big| \\ &\quad \leq \; c\, 2^{(j+j')}(\max\{diam\ \tau_y, diam\ \tau_y\})^{D-d}(diam\ \tau_y)^2(diam\ \tau_y)^2 \cdot \\ &\qquad\qquad \cdot dist\ (\kappa_i(\tau_y), \kappa_l(\tau_x))^{-2-r-D+d} \end{aligned}$$

is valid provided that $2 + r + D - d > 0$.

The principal strategy is to choose the diameter of the sub-domains proportional to the distance from the singularity while the degree D has to be adapted to maintain the desired accuracy. That is, we proceed as follows:

- When integrating over domains $\tau_x \times \tau_y$ where τ_x and τ_y share an edge, a vertex or are identical, then in general the integral is singular. In this case some sort of *regularization* should be applied to reduce the integral to a weakly singular integral [N, PS]. Then we use transformation techniques like Duffy's trick proposed in [S] to end up with analytical integrals [PS].

- For the remaining integrals we apply the following adaptive quadrature method. We divide the domain of integration into sub-domains of different sizes. On each sub-domain we apply e.g. a product-type Gaussian quadrature of variable degree D. Without loss of generality we assume that $J \geq j \geq j'$ and choose a parameter $q > 1$.

  If the domains $\mathcal{T}^j_x$, $\mathcal{T}^{j'}_y$ satisfy the estimate

  $$\text{dist}\ (\widehat{\mathcal{T}}^j, \widehat{\mathcal{T}}^{j'}) \geq q \max(\text{diam}\ (\mathcal{T}^j), \text{diam}\ (\mathcal{T}^{j'})), \tag{4.25}$$

  we employ Gaussian quadrature with order

  $$D = \alpha[(\min\{j', j\} - J + \alpha^{-1})r + (J - j)d' + (J - j')d' + \log_2 \epsilon]_+$$

  with $d < d'$, $\alpha \sim (\log_2 q)^{-1}$ and $[u]_+ := \max\{0, u\}$.

  Otherwise, as long as $j > j'$, we continue by subdividing the domain $\mathcal{T}^{j'}$ into four sub-domains $\tau^{\lambda'}_{\nu'}$ with $\lambda' = j' + 1$. We repeat the procedure for $\mathcal{T}^j$ and all sub-domains $\tau^{\lambda'}_{\nu'}$ of $\mathcal{T}^{j'}$ until we arrive at

  $$\text{dist}\ (\widehat{\mathcal{T}}^j, \widehat{\tau}^{\lambda'}_{\nu'}) \geq q\ \max(\text{diam}\ (\mathcal{T}^j), \text{diam}\ (\tau^{\lambda'}_{\nu'})). \tag{4.26}$$

  Then we apply numerical integration with order

  $$\begin{aligned} D \;=\; &[\alpha(\min\{j', j\} - J + \alpha^{-1})r \\ &+ (J - \min\{j, j'\})d' + (J - \max\{j, j'\})d' + \log_2 \epsilon) - 2]_+. \end{aligned}$$

If $j = j'$ or $j = \lambda'$, there are two possibilities. In case

$$\mathcal{T}^j \cap \tau_{\nu'}^{\lambda'} \neq \emptyset \tag{4.27}$$

we have to approximate a singular integral as mentioned above.

If finally neither condition (4.25) nor condition (4.27) applies and $j = \lambda'$, we divide both $\mathcal{T}^j$ and $\tau_{\nu'}^{\lambda'}$ and repeat the same procedure for each sub-domain τ_ν^λ of $\mathcal{T}^j$ and $\tau_{\nu'}^\lambda$ of $\mathcal{T}^{j'}$, respectively, until τ_ν^λ and $\tau_{\nu'}^\lambda$ have a distance proportional to $2^{-\lambda}$.

In case of Lipschitz domains the above algorithm terminates after a finite number of subdivisions. Once a matrix entry has been decided to be computed we apply the above quadrature procedure yielding an approximate compressed matrix $\mathbf{A}_{\Psi_J}^{CQ}$.

We remark that for arbitrary $J \in \mathbb{N}$ we may need arbitrarily high accuracy. For that reason we have assumed analyticity of the surface elements Γ_i.

If we assume that evaluating the kernel function at one quadrature point requires $\mathcal{O}(J^a)$, $a \geq 1$, floating point operations, we compute the complete fully discretized compressed matrix $\mathbf{A}_{\Psi_J}^{CQ}$ with $\mathcal{O}(2^{2J}) = \mathcal{O}(\dim S_J)$ floating point operations, i.e., with linear complexity, while still preserving the accuracy of the Galerkin scheme.

Theorem 4.3 *Let the assumptions in Theorem* 4.2 *be satisfied. Then the fully discretized compressed system* $\mathbf{A}_{\Psi_J}^{CQ}\mathbf{u}_J^{CQ} = \mathbf{f}_J$ *possesses a unique solution* u_J^{CQ} *realizing asymptotically optimal accuracy*

$$\left\| u - u_J^{CQ} \right\|_{H^t} \lesssim 2^{j(t-s)} \|u\|_{H^s} \tag{4.28}$$

where $-d + r \leq t < \gamma$, $t \leq s$, $\frac{r}{2} \leq s \leq d$ *and* u *is the exact solution of* $Au = f$. *Moreover, the nonzero coefficients of the matrix* $\mathbf{A}_{\Psi_J}^{CQ}$ *can be computed at the expense of* $\mathcal{O}(\dim S_J)$ *floating point operations.*

Summarizing our conclusions, we compute an approximate solution of an integral equation in linear time, requiring an amount of storage which remains proportional to the number of unknowns $\dim S_J$. This can be performed without diminishing the accuracy of the full Galerkin method. Conversely, if the order of exactness of the trial spaces is d, in order to achieve optimal accuracy

$$\|u - u_J^{CQ}\|_{-d+r} \leq \epsilon$$

for a given $\epsilon > 0$ we need $\dim S_J = \mathcal{O}(\epsilon^{-2/(2d-r)})$. Here we have proposed a method to achieve this accuracy with only $\mathcal{O}(\epsilon^{-2/(2d-r)})$ floating point operations.

5 Construction of Bases

In the above analysis we have assumed a number of concrete properties of multiscale bases which are crucial for the success of the outlined techniques. The actual construction of such bases is in general by no means a trivial task. This section is to review some recent developments in this regard, in particular, in connection with (II), (III).

5.1 The Shift-Invariant Case

We begin with a few comments on the classical setting of wavelets defined on the Euclidean space $\mathbb{R}^n$. Although this setting is obviously not suited for applications of the above type some of its ingredients do turn out to be useful.

A function $\theta \in L_2(\mathbb{R}^n)$ is called *refinable* if there exists a *mask* $\mathbf{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}^n}$ such that

$$\theta(x) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \theta(2x - \alpha). \tag{5.1}$$

More generally, one could consider scaling by powers of some matrix M whose eigenvalues are all larger than one or vector valued versions of the form

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}} \mathbf{a}_\alpha \Phi(Mx - \alpha)$$

where now $\Phi(x) = (\theta_1(x), \ldots, \theta_N(x))^T$ and the $\mathbf{a}_\alpha$ are $N \times N$ matrices (see e.g. [DM2]). For our purposes it will suffice to stick with (5.1) for the special univariate case $n = 1$. In this case the concept of biorthogonal wavelets is well developed. To describe this, we call two refinable functions θ, $\tilde{\theta}$ a *dual pair* if

$$\left\langle \theta, \tilde{\theta}(\cdot - k) \right\rangle_{\mathbb{R}} := \int_{\mathbb{R}} \theta(x) \overline{\tilde{\theta}(x-k)} \, \mathrm{d}x = \delta_{0,k}, \quad k \in \mathbb{R}. \tag{5.2}$$

We will confine the discussion to the case that both θ and $\tilde{\theta}$ have compact support contained in $[-\ell, \ell]$. It is well-known that θ and $\tilde{\theta}$ can then be normalized to satisfy $\int_{\mathbb{R}} \theta(x) \, \mathrm{d}x = \int_{\mathbb{R}} \tilde{\theta}(x) \, \mathrm{d}x = 1$ which means that

$$1 = \sum_{k \in \mathbb{Z}} \theta(x - k) = \sum_{k \in \mathbb{Z}} \tilde{\theta}(x - k). \tag{5.3}$$

One easily derives from (5.2) that the collections $\Theta_j := \{2^{j/2}\theta(2^j \cdot - k) : k \in \mathbb{Z}\}$ and analogously $\tilde{\Theta}_j$ are uniformly stable in the sense of (3.3) while refinability ensures that

$$S(\Theta_j) \subset S(\Theta_{j+1}), \quad S(\tilde{\Theta}_j) \subset S(\tilde{\Theta}_{j+1}), \quad j \in \mathbb{Z}. \tag{5.4}$$

In this case one can identify appropriate complement spaces as follows [CDF]. Let

$$\omega(x) := \sum_{k \in \mathbb{Z}} (-1)^k \tilde{a}_{1-k} \theta(2x - k), \quad \tilde{\omega}(x) := \sum_{k \in \mathbb{Z}} (-1)^k a_{1-k} \tilde{\theta}(2x - k). \tag{5.5}$$

One verifies that the functions $\omega_{j,k} := 2^{j/2}\omega(2^j \cdot - k)$, $\tilde{\omega}_{j,k} := 2^{j/2}\tilde{\omega}(2^j \cdot - k)$, $j, k \in \mathbb{Z}$, form biorthogonal collections,

$$\langle \omega_{j,k}, \tilde{\omega}_{j',k'} \rangle_{\mathbb{R}} = \delta_{(j,k),(j',k')}, \quad j, j', k, k' \in \mathbb{Z}. \tag{5.6}$$

Moreover, $\{\omega_{j,k}\}_{j,k \in \mathbb{Z}}$, $\{\tilde{\omega}_{j,k}\}_{j,k \in \mathbb{Z}}$ are Riesz bases [CDF]. The classical derivation uses Fourier analysis [CD, CDF, Vi]. It can also be deduced with the aid of Theorems 3.1 or 3.2 as follows. One concludes from (5.3) that (3.23) holds for $s \leq 1$. Moreover, it is known that $\theta, \tilde{\theta} \in L_2(\mathbb{R})$ actually implies $\theta, \tilde{\theta} \in H^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$ [CD, Vi] and that this confirms the validity of (3.24) for $s < \varepsilon$. So the claim follows from Theorem 3.2 for $s = 0$. We could have also applied Theorem 3.1 where $\omega = \omega_1$ can be chosen as the first order L_2-modulus of smoothness [D3].

5.2 Wavelets on the Intervall $[0,1]$

One can use next the shift-invariant case to construct wavelets on an interval which we may choose for simplicity to be $[0,1]$. Simply restricting the translates $\theta(2^j\cdot-k)$, $\tilde\theta(2^j\cdot-k)$ to $[0,1]$ would destroy the biorthogonality relations as well as stability properties. The key idea is to modify those translates which interfere with the ends of the interval. Since this will be used several times we explain it in a little more detail. Suppose that θ and $\tilde\theta$ are exact of order d, $\tilde d$, respectively, i.e.,

$$x^r = \begin{cases} \sum\limits_{k\in\mathbb{Z}} \left\langle (\cdot)^r, \tilde\theta(\cdot-k)\right\rangle_{I\!R} \theta(x-k), & 0\le r\le d-1,\\ \sum\limits_{k\in\mathbb{Z}} \langle (\cdot)^r, \theta(\cdot-k)\rangle_{I\!R}\, \tilde\theta(x-k), & 0\le r\le \tilde d-1, \end{cases} \tag{5.7}$$

where $\tilde d\ge d\ge 2$. It is known that this is equivalent to the fact that the *symbols*

$$a(z) := \sum_{k\in\mathbb{Z}} a_k z^k, \quad \tilde a(z) := \sum_{k\in\mathbb{Z}} \tilde a_k z^k,$$

contain $(1+z)^d$, $(1+z)^{\tilde d}$ as a factor, respectively. The usual approach taken in [CDV, AHJP, CQ] is to introduce for $k=0,\dots,d-1$ boundary near functions

$$\begin{aligned} \theta^L_{j,L-d+k} &:= \sum_{m=-\infty}^{L-1} \alpha^L_{j,m,k} 2^{j/2}\theta(2^j\cdot-m)|_{[0,1]},\\ \theta^R_{j,2^j-R+d-k} &:= \sum_{m=2^j-R+1}^{\infty} \alpha^R_{j,m,k} 2^{j/2}\theta(2^j\cdot-m)|_{[0,1]}, \end{aligned} \tag{5.8}$$

where the

$$\alpha^L_{j,m,k} = \left\langle (2^j\cdot)^k, 2^{j/2}\tilde\theta(2^j\cdot-m)\right\rangle_{I\!R} = \int_{I\!R} x^k\overline{\tilde\theta(x-k)}\,\mathrm{d}x$$

are actually independent of j. The $\alpha^R_{j,m,k}$ are defined analogously, so that when $L=R$ and θ, $\tilde\theta$ even,

$$\theta^L_{j,k} = \theta^R_{j,2^j-k-R+d}(1-\cdot), \quad k=0,\dots,d-1. \tag{5.9}$$

Setting $\Theta_j = \{\theta^L_{j,k}, \theta^R_{j,2^j-R+d-k} : k=0,\dots,d-1\}\cup\{2^{j/2}\theta(2^j\cdot-k) : k=L,\dots,2^j-R\}$ the construction ensures that

$$\Pi_d([0,1])\subset S(\Theta_j), \qquad 2^{j-1} > \ell+\max\{L,R\}+1, \tag{5.10}$$

which, in turn, is known to imply (3.23) for $S(\Theta_j)=V_j$ and $s\le d$. The common approach is to choose L, R large enough so that one can form biorthogonal systems $\tilde\Phi_j$ from (unchanged) translates $\tilde\theta(2^j\cdot-k)$ supported strictly in the interior of $[0,1]$. While this simplifies biorthogonalization of the boundary near functions the dual multiresolution $\tilde{\mathcal{S}}$ looses its exactness. Since the exactness of $\tilde{\mathcal{S}}$ will matter in our applications we follow a different line, defining also modified functions $\tilde\theta^L_{j,k}$, $\tilde\theta^R_{j,k}$ on the dual side, so as to preserve the order $\tilde d$ of exactness for $\tilde{\mathcal{S}}$ as well. When θ is a cardinal B-spline with integer knots and $\tilde\theta$ any of the duals derived in [CDF] it can be proved that the boundary near functions can

always be biorthogonalized [DKU2] so that now also $\tilde{\mathcal{S}}$ is exact of order $\tilde{d}$ and (3.23) holds. Forming corresponding biorthogonal wavelets by making similar modifications near the ends of the interval as in [CDV, AHJP] or using techniques detailed below in Section 5.5, we obtain again biorthogonal bases $\{\omega_{j,k}\}_{j,k\in\mathbb{Z}}$, $\{\tilde{\omega}_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L_2([0,1])$. While Fourier techniques can no longer be used to ensure the Riesz basis property it follows again from Theorem 3.2 that $\{\omega_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\tilde{\omega}_{j,k}\}_{j,k\in\mathbb{Z}}$ are indeed Riesz bases for $L_2[0,1]$. We will exploit these facts in the following two important cases.

5.3 The Ladyšenskaja–Babuška–Brezzi Condition

We return to problem (III) concerning the construction of compatible pairs of trial functions for treating problems of the type (2.6) which may be formulated in general as follows. Given a symmetric and continuous bilinear form $a : V \times V \to \mathbb{R}$ and a continuous bilinear form $b : V \times M \to \mathbb{R}$ where V and M are Hilbert spaces. For $f \in V'$ (relative to a dual form $\langle\cdot,\cdot\rangle$) find $(u,p) \in V \times M$ such that

$$\begin{array}{lcll} a(u,v) + b(v,p) & = & \langle f, v\rangle\,, & v \in V, \\ b(u,q) & = & 0, & q \in M. \end{array} \tag{5.11}$$

It is well-known that (5.11) possesses a unique solution if and only if a is V-elliptic on $\{v \in V : b(v,q) = 0,\ q \in M\}$ and if the inf-sup condition

$$\inf_{q\in M}\sup_{v\in V}\frac{b(v,q)}{\|v\|_V\,\|q\|_M} \geq \beta > 0 \tag{5.12}$$

holds. As explained in (II) we seek for pairs of finite dimensional spaces $V_j \subset V$, $M_j \subset M$ such that (5.12) holds uniformly in $j \in \mathbb{N}_0$ (see (2.8)). Our starting point is the following well-known observation due to Fortin [BF, GR].

Proposition 5.1 *Suppose that V, M satisfy* (5.12). *Then* (2.8) *holds uniformly in j if and only if there exist linear operators $Q_j : V \to V_j$ satisfying*

$$\|Q_j v\|_V \lesssim \|v\|_V\,, \quad v \in V, \tag{5.13}$$

and

$$b(v - Q_j v, q_j) = 0, \quad v \in V,\ q_j \in M_j. \tag{5.14}$$

According to (II) we are interested in the case

$$b(v,q) = (\operatorname{div} v, q)_\Omega := \int_\Omega \operatorname{div} v(x) q(x)\,\mathrm{d}x$$

where in the sequel we restrict the discussion for simplicity to $\Omega = [0,1]^n$. Since (5.14) may be viewed as a biorthogonality condition we wish to construct a suitable sequence $\mathcal{Q} = \{Q_j\}$ of operators satisfying (5.13) and (5.14) with the aid of techniques from Sections 5.1 and 5.2.

To this end, note that biorthogonality can be expressed in terms of the respective symbols, namely, when θ, $\tilde{\theta}$ form a dual pair (5.2) then

$$a(z)\overline{\tilde{a}(z)} + a(-z)\overline{\tilde{a}(-z)} = 4. \tag{5.15}$$

Moreover, if the modified symbols

$$a^*(z) := \frac{2}{1+z}a(z), \quad \tilde{a}^*(z) := \frac{1+\overline{z}}{2}\tilde{a}(z) \tag{5.16}$$

correspond to refinable functions θ^*, $\tilde{\theta}^*$ (which is the case e.g. when $\theta \in H^1(\mathbb{R})$) then θ^*, $\tilde{\theta}^*$ still form a dual pair. Roughly speaking, θ^*, $\tilde{\theta}^*$ are obtained from θ, $\tilde{\theta}$ by differentiation and integration. The precise circumstances are described by the following observation which is essentially due to Lemarié [L1, L2, DKU1].

Lemma 5.1 *Let $\theta, \tilde{\theta} \in L_2(\mathbb{R})$ be a dual pair of compactly supported refinable functions with masks $\mathbf{a}^0$, $\tilde{\mathbf{a}}^0$, respectively, and normalized as in (5.3). If $\theta \in H^1(\mathbb{R})$ then there exists a dual pair θ^*, $\tilde{\theta}^*$ of compactly supported refinable functions in $L_2(\mathbb{R})$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \theta^*(x) - \theta^*(x-1), \quad \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}^*(x) = \tilde{\theta}(x+1) - \tilde{\theta}(x) \tag{5.17}$$

holds. Their symbols $\mathbf{a}^{(0,*)}$, $\tilde{\mathbf{a}}^{(0,*)}$ are given by (5.16) and the corresponding biorthogonal wavelets ω^*, $\tilde{\omega}^*$ (5.5) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\omega(x) = 4\omega^*(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\omega}^*(x) = -4\tilde{\omega}(x). \tag{5.18}$$

Defining now for simplicity

$$\omega_0 := \theta, \quad \omega_1 := \omega, \quad \tilde{\omega}_0 := \tilde{\theta}, \quad \tilde{\omega}_1 := \tilde{\omega}$$

let for $i = 1, \ldots, n$, $e \in \{0,1\}^n =: E$,

$$\psi_e := \prod_{i=1}^{n} \omega_{e_i}, \quad \tilde{\psi}_e := \prod_{i=1}^{n} \tilde{\omega}_{e_i}, \tag{5.19}$$

as well as

$$\psi_e^{(i)} := \left(\prod_{r=1,\ldots,n,\ r\neq i} \omega_{e_r} \right) \omega_{e_i}^*, \quad \tilde{\psi}_e^{(i)} := \left(\prod_{r=1,\ldots,n,\ r\neq i} \tilde{\omega}_{e_r} \right) \tilde{\omega}_{e_i}^* \tag{5.20}$$

and finally

$$\varphi = \psi_0, \quad \varphi^{(i)} = \psi_0^{(i)}, \quad \tilde{\varphi} = \tilde{\psi}_0, \quad \tilde{\varphi}^{(i)} = \tilde{\psi}_0^{(i)}. \tag{5.21}$$

Clearly, $(\varphi, \tilde{\varphi})$, $(\varphi^{(i)}, \tilde{\varphi}^{(i)})$ form again dual pairs in $L_2(\mathbb{R}^n)$.

We have collected now all prerequisites to construct compatible pairs of trial spaces V_j, M_j satisfying the LBB condition first for $\Omega = \mathbb{R}^n$ and $b(u,p) = (\operatorname{div} u, p)_{\mathbb{R}^n}$. In fact, setting

$$\psi_{j,k} = 2^{nj/2}\psi_e(2^j \cdot - \alpha), \quad k = (e,\alpha) \in E \times \mathbb{Z}^n,$$

and analogously $\tilde{\psi}_{j,k}$, $\psi^{(i)}_{j,k}$, $\tilde{\psi}^{(i)}_{j,k}$, $\varphi_{j,k}$, $\varphi^{(i)}_{j,k}$, $\tilde{\varphi}_{j,k}$, $\tilde{\varphi}^{(i)}_{j,k}$, we obtain pairs of biorthogonal bases

$$\Phi_j := \{\varphi_{j,k} \,:\, k \in \{0\} \times \mathbb{Z}^n\}, \quad \tilde{\Phi}_j := \{\tilde{\varphi}_{j,k} \,:\, k \in \{0\} \times \mathbb{Z}^n\}$$

and likewise $\Phi^{(i)}_j$, $\tilde{\Phi}^{(i)}_j$, $i = 1, \ldots, n$. The corresponding complement bases are denoted by Ψ_j, $\tilde{\Psi}_j$, $\Psi^{(i)}_j$, $\tilde{\Psi}^{(i)}_j$. Now let

$$M_{j+1} := S(\Phi_j), \quad V_j := S(\tilde{\Phi}^{(1)}_j) \times \cdots \times S(\tilde{\Phi}^{(n)}_j). \tag{5.22}$$

In view of the above biorthogonality relations the mappings $Q_j : H^1_0(\Omega)^n \to V_j$ defined by

$$(Q_j v)_i = \sum_{k \in \{0\} \times \mathbb{Z}^n} (v_i, \varphi^{(i)}_{j,k}) \tilde{\varphi}^{(i)}_{j,k} \tag{5.23}$$

are projectors and for $i = 1, \ldots, n$ one has

$$(v - Q_j v)_i = \sum_{m=j}^{\infty} \sum_{k \in E_* \times \mathbb{Z}^n} (v_i, \psi^{(i)}_{m,k}) \tilde{\psi}^{(i)}_{m,k} \tag{5.24}$$

where $E_* = E \setminus \{0\}$. But from the construction of the $\psi^{(i)}_{m,k}$, $\tilde{\psi}^{(i)}_{m,k}$ and Lemma 5.1 one easily deduces that

$$\frac{\partial}{\partial x_i}(v - Q_j v)_i = \sum_{m=j}^{\infty} \sum_{k \in E_* \times \mathbb{Z}^n} d_{m,k} \tilde{\psi}_{m,k} \in \bigoplus_{m=j}^{\infty} S(\tilde{\Psi}_m) \tag{5.25}$$

and thus

$$\operatorname{div}(v - Q_j v) \in \bigoplus_{m=j}^{\infty} S(\tilde{\Psi}_m). \tag{5.26}$$

Biorthogonality ensures that

$$S(\Phi_{j-1}) \perp S(\tilde{\Psi}_m), \quad m \geq j,$$

so that one has

$$b(v - Q_j v, q) = 0, \quad q \in M_j. \tag{5.27}$$

Since for sufficiently regular $\tilde{\varphi}^{(i)}$ the projectors are easily seen to be H^1-bounded [D2], the hypotheses of Proposition 5.1 are satisfied so that the above pairs M_j, V_j fulfill the LBB condition.

The next step is to extend these constructions to bounded domains. For $\Omega = [0,1]^n$ one can use the techniques described in Section 5.2 to adapt multiresolution analyses based on θ, $\tilde{\theta}$, θ^*, $\tilde{\theta}^*$ to $[0,1]$ and then form tensor products as above. The crucial point is that the relation

$$S\Big(\frac{\partial}{\partial x_i} \tilde{\Psi}^{(i)}_j\Big) \subseteq S(\tilde{\Phi}_j), \quad i = 1, \ldots, n, \tag{5.28}$$

can be shown to persist under the modifications of the boundary near functions. Therefore the arguments (5.25), (5.26) remain valid so that Proposition 5.1 confirms again the validity of the LBB condition for the resulting spaces M_j, V_j on $\Omega = [0,1]^n$ [DKU1].

Choosing θ as a B-spline and $\tilde{\theta}$ as a dual derived in [CDF] a whole family of spaces M_j, V_j is constructed in [DKU1] satisfying the LBB condition as well as direct and inverse estimates of the form (3.23) and (3.24) for ranges of s depending on the regularity and exactness of θ and $\tilde{\theta}$. In particular, the Q_j defined in (5.23) give rise to norm equivalences of the form (3.25) so that the above construction not only yields admissible trial spaces but also lays the foundation for efficient solvers of the resulting linear systems (2.7). It is shown in [DKU1] that preconditioners based on the change of bases as described in Section 4.1 can be applied to $\mathbf{A}_j = \mathbf{A}_h$ in (2.7) as well as to the Schur complements $\mathbf{B}_j\mathbf{A}_j^{-1}\mathbf{B}_j^T$ (when dealing with fully implicit time discretizations in the time dependent case (2.5)) to guarantee that iterative schemes of the type discussed in [BP] are asymptotically optimal. For details the reader is referred to [DKU1].

5.4 Computational Aspects

Working with bases which result from modifying a multiresolution analysis for $\mathbb{R}^n$ offers some interesting computational advantages. In fact, there is an essentially dimension independent unified platform for computing quantities involving integrals of products of derivatives of refinable functions and wavelets. More precisely, suppose that $\varphi^i \in L_2(\mathbb{R}^n)$, $i = 0, \ldots, m$, are (possibly) different compactly supported refinable functions with refinement masks $\mathbf{a}^i$ (see (5.1)). It turns out that quantities of the form

$$\int_{\mathbb{R}^n} \varphi^0(x)(D^{\mu^1})\varphi^1(x-\alpha^1)\cdots(D^{\mu^m}\varphi^m)(x-\alpha^m)\,\mathrm{d}x =: I(\alpha^1,\ldots,\alpha^m) \tag{5.29}$$

where $D^\nu := \prod_{l=1}^n \frac{\partial^{\nu_l}}{(\partial x_l)^{\nu_l}}$ can be computed exactly (up to round off) without resorting to quadrature [DM1]. Before describing this in more detail let us briefly pause to indicate the usefulness and consequences of this fact in the context of the above applications.

The entries of the matrices arising in problems of type (I), (II) on $\Omega \subset \mathbb{R}^n$ are of the form

$$\int_\Omega c_{\eta\beta}(x)(D^\eta\varphi^1)(2^Jx-\alpha^1)(D^\eta\varphi^2)(2^Jx-\alpha^2)\,\mathrm{d}x \tag{5.30}$$

where $c_{\eta\beta}(x)$ are some possibly variable coefficients and $\eta, \beta, \alpha^1, \alpha^2 \in \mathbb{Z}^n$. Since in general the setup of a linear system (4.2) takes up a major part of all computational costs, it is important to note that, in view of (4.8), the use of multiscale bases only requires to determine the quantities (5.30) on the finest discretization level J. One would here replace the nonconstant coefficients $c_{\eta\beta}(x)$ by an expression of the form

$$\sum_{\alpha^3\in\mathbb{Z}^n} \gamma^3(c_{\eta\beta})\,\varphi^3(2^Jx-\alpha^3) \tag{5.31}$$

where γ^3 are e.g. dual functionals for another compactly supported refinable function φ^3 with mask $\mathbf{a}^3$. In particular, φ^3 could be the characteristic function so that nonsmooth coefficients are also covered. In the simplest case $\Omega = [0,1]^n$ the integrals (5.30) can be written as a sum of integrals over all cubes of diameter $\sim 2^{-J}$ which in turn are

integrals over $\mathbb{R}^n$ by introducing the characteristic function φ^0 over the microcubes as an additional factor. Since now all involved functions live with respect to level J, the substitution $2^J x \to x$ readily yields terms of the form (5.29). Note that the accuracy of the computations depends only on the quality of approximation (5.31) to $c_{\eta\beta}(x)$ when the integrals (5.29) are computed exactly. Thus, no more than four factors appear in the latter integral to model the effects of variable coefficients and the bounded domain. The right hand sides of systems (4.2) and (2.7) can be handled in the same fashion. Due to the nature of their entries, the involved integrals only contain at most three factors and no derivatives. Such computations and the setup of the linear system with polynomial coefficients for $n = 1$ can e.g. be found in [K1].

The first step towards the evaluation of (5.29) is the observation that the function $I(\alpha) = I(\alpha^1, \ldots, \alpha^m)$ defined on $\mathbb{R}^{mn}$ is also compactly supported and refinable,

$$2^{-|\mu|} I(x) = \sum_{\alpha \in \mathbb{Z}^{nm}} c_\alpha \, I(2x - \alpha), \qquad x \in \mathbb{R}^{mn}, \tag{5.32}$$

since the φ^i, $i = 0, \ldots, m$, are, with mask coefficients given by

$$c_\mu := 2^{-n} \sum_{\nu \in \mathbb{Z}^n} a^0_\nu \prod_{i=1}^{m} a^i_{\nu - \mu^i}, \quad \mu \in \mathbb{Z}^{mn}. \tag{5.33}$$

Here $|\mu| = |\mu^1| + \cdots + |\mu^m|$ is the total number of derivatives in (5.29). Because of the validity of (5.32) we call I a *refinable integral*. The presence of any such refinement equation (5.32) has two computational advantages:

- If I is known at all lattice points $\mathbb{Z}^{mn}$, I can be determined in linear time at all dyadic points $\mathbb{Z}^{mn}/2$, $\mathbb{Z}^{mn}/4, \ldots$ by successively using the refinement equation.
- To compute I on $\mathbb{Z}^{mn}$, rewrite by a change of indices (5.32) as

$$2^{-|\mu|} I(\alpha) = \sum_{\nu \in \mathbb{Z}^{nm}} c_{2\alpha - \nu} \, I(\nu), \qquad \alpha \in \mathbb{Z}^{mn}. \tag{5.34}$$

Since due to the finite support of the masks only finitely many $I(\alpha)$ are different from zero (5.34) can be interpreted as the problem of computing an *eigenvector* of finite length for the eigenvalue $2^{-|\mu|}$. These facts have been already observed in [CDM]. They have also been utilized in [LRT] for the purpose of evaluating univariate refinable functions and also integrals of the form (5.29) in one dimension (called connection coefficients there). However, in order to make this approach work one has to deal with the following uniqueness questions. When $|\mu| = 0$ one has to make sure that one is a simple eigenvalue. When $|\mu| > 0$ and $n > 1$ the corresponding refinable integrals already give rise to different eigenvectors. So one has to find additional conditions which uniquely determine these eigenvectors. These problems have been addressed and solved first in [DM1]. One of the conditions derived there is based on certain factorizations of the symbols $a^i(z)$. In the multivariate case these factorizations may not always exist. Alternative conditions which are always applicable can be derived with the aid of subdivision techniques.

For a finitely supported mask $\mathbf{a}$ on $\mathbb{Z}^{mn}$, the *stationary subdivision scheme*

$$(S_{\mathbf{a}} \lambda)_\alpha := \sum_{\beta \in \mathbb{Z}^{mn}} a_{\alpha - 2\beta} \lambda_\beta, \qquad \alpha \in \mathbb{Z}^{mn},$$

is said to *converge* if for any $\lambda \in \ell_\infty(\mathbb{Z}^{mn})$ there is some $f_\lambda \in \mathcal{C}(\mathbb{R}^{mn})$ such that

$$\lim_{k\to\infty} \left(\sup_{\alpha\in\mathbb{Z}^{mn}} \left| (S_{\mathbf{a}}^k \lambda)_\alpha - f_\lambda\left(\frac{\alpha}{2^k}\right) \right| \right) = 0.$$

The relevance of these notions for the present context is indicated by the following facts. In [CDM] it is shown that convergence of $S_{\mathbf{a}}$ implies the existence of a unique $\varphi \in \mathcal{C}_0(\mathbb{R}^{mn})$ which is refinable with respect to $\mathbf{a}$. Conversely, $\mathbf{a}$-refinability and ℓ_∞-stability of $\varphi \in \mathcal{C}_0(\mathbb{R}^{mn})$ imply convergence in the above sense. A refined analysis of convergence based on asymptotic expansions is the heart of the proof of the following result. It identifies additional moment conditions which guarantee uniqueness of the eigenvector I in (5.34).

Theorem 5.1 *Let $\varphi^i \in \mathcal{C}_0^\sigma(\mathbb{R}^n)$ be $\mathbf{a}^i$-refinable and ℓ_∞-stable for each $i = 0, \ldots, m$. Then for every $\mu \in \mathbb{Z}_+^{mn}$, $|\mu| \le \sigma$, there exists a unique finitely supported sequence $\{W_\beta : \beta \in \mathbb{Z}^{mn}\}$ such that*

$$\begin{aligned} \sum_{\beta\in\mathbb{Z}^{mn}} c_{2\alpha-\beta} W_\beta &= 2^{-|\mu|} W_\alpha, \qquad \alpha \in \mathbb{Z}^{mn}, \\ \sum_{\alpha\in\mathbb{Z}^{mn}} (-\alpha)^\nu \, W_\alpha &= \mu!\, \delta_{\nu,\mu}, \qquad |\nu| \le |\mu|, \quad \nu, \mu \in \mathbb{Z}_+^{mn}. \end{aligned} \tag{5.35}$$

The refinable integrals are given by

$$I(\alpha) = (-1)^{|\mu|} W_\alpha, \qquad \alpha \in \mathbb{Z}^{mn}. \tag{5.36}$$

The assertion of the theorem actually remains true under weaker smoothness assumptions. For instance, one can take $\varphi^0 = \chi_{[0,1)^n}$ and φ^i any μ^i-times continuously differentiable refinable function, $i = 1, \ldots, m$, when integrals with derivatives of order μ^i in (5.29) are to be computed. If one wishes to evaluate any multivariate function I satisfying a refinement equation (5.32) with mask coefficients $\mathbf{c}$, i.e. $|\mu| = 0$, or compute its derivatives at lattice points, the solution W_α (5.36) already gives the desired quantities. It should also be mentioned that Theorem 5.1 applies e.g. to cardinal B-Splines and box splines [BHR].

The procedure for computing the refinable integrals (5.29) based on the above theorem is implemented in C++-routines for up to four factors in the integral in dimensions $n \le 2$ and up to three in 3D [K2]. Of course, if the functions φ^i are tensor products of univariate functions as in Section 5.2 the integrals factor into univariate refinable integrals so that also four factors in three dimensions can be handled efficiently. As input, the software only requires the mask coefficients $\mathbf{a}^i$, $i = 0, \ldots, m$, and information about their supports. The system (5.35) is then set up with dynamical storage allocation and is solved in the current version by computing the QR factorization of the system matrix.

5.5 Stable Completions

The explicit computability of biorthogonal complement bases in $L_2(\mathbb{R}^n)$ rests to a great deal on the possibility of reducing the computation to manipulating Laurent polynomials (see (5.15), (5.16)). As soon as one has to deal with function spaces defined on domains other than $\mathbb{R}^n$ or the torus, such techniques are usually not applicable. The common

theme of [D1, D2, D3, CDP] is to develop an appropriate general framework for multiscale techniques which are applicable to more realistic problems. Two major issues arise in this context. On one hand, one has to bring out the relevant facts pertaining to stability properties and norm equivalences as indicated in Section 3 above. On the other hand, one has to develop new tools for actually realizing these properties for concrete constructions. We briefly review next one general concept that has proven to be quite useful in several applications [CDP].

To describe this for the general setting in Section 3, let us first note that the nestedness of the spaces $S(\Phi_j)$ and (uniform) stability of the bases Φ_j imply the existence of refinement matrices $\mathbf{M}_{j,0} = (m^j_{\ell,k})_{\ell\in\Delta_{j+1},k\in\Delta_j}$ representing (uniformly) bounded mappings from $\ell_2(\Delta_j)$ into $\ell_2(\Delta_{j+1})$ such that

$$\varphi_{j,k} = \sum_{\ell\in\Delta_{j+1}} m^j_{\ell,k}\varphi_{j+1,\ell}, \quad k\in\Delta_j, \tag{5.37}$$

where whenever in the following infinite sums occur convergence is to be understood in the absolute sense. It will be convenient to view Φ_j formally as a (column) vector so that (5.37) reads

$$\Phi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,0} \tag{5.38}$$

and $\mathbf{M}_{j,0}\in[\ell(\Delta_j),\ell(\Delta_{j+1})]$, the space of bounded linear mappings from $\ell_2(\Delta_j)$ into $\ell_2(\Delta_{j+1})$. Next one observes that any collection $\Psi_j\subset S(\Phi_{j+1})$ spans a complement of $S(\Phi_j)$ in $S(\Phi_{j+1})$ such that $\{\Phi_j\cup\Psi_j\}$ is uniformly stable if and only if Ψ_j has the form

$$\Psi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,1} \tag{5.39}$$

for some $\mathbf{M}_{j,1}\in[\ell_2(\nabla_j),\ell_2(\Delta_{j+1})]$ such that $\mathbf{M}_j = (\mathbf{M}_{j,0},\mathbf{M}_{j,1})\in[\ell_2(\Delta_j\cup\nabla_j),\ell_2(\Delta_{j+1})]$ is uniformly bounded and boundedly invertible. That is, there exists $\mathbf{G}_j\in[\ell_2(\Delta_{j+1}),\ell_2(\Delta_j\cup\nabla_j)]$ blocked as $\begin{pmatrix}\mathbf{G}_{j,0}\\ \mathbf{G}_{j,1}\end{pmatrix}$ such that

$$\mathbf{G}_j\mathbf{M}_j = \mathbf{M}_j\mathbf{G}_j = \mathbf{I}, \quad \|\mathbf{M}_j\|, \|\mathbf{G}_j\| = \mathcal{O}(1), \quad j\in I\!N_0, \tag{5.40}$$

i.e.,

$$\mathbf{G}_{j,i}\mathbf{M}_{j,i'} = \delta_{i,i'}\mathbf{I}, \quad \mathbf{M}_{j,0}\mathbf{G}_{j,0} + \mathbf{M}_{j,1}\mathbf{G}_{j,1} = \mathbf{I} \tag{5.41}$$

and

$$\Phi_{j+1}^T = \Phi_j^T\mathbf{G}_{j,0} + \Psi_j^T\mathbf{G}_{j,1}. \tag{5.42}$$

This tells us how the single scale basis functions on the fine level $j+1$ are reconstructed from the coarse scale basis functions on level j and the complement basis Ψ_j. It is not hard to verify that the multiscale transformation $\mathbf{T}_j$ from (3.9) has the form

$$\mathbf{T}_j = \hat{\mathbf{T}}_{j-1}\cdots\hat{\mathbf{T}}_0, \quad \hat{\mathbf{T}}_\ell = \begin{pmatrix}\mathbf{M}_\ell & 0\\ 0 & \mathbf{I}\end{pmatrix}.$$

Any $\mathbf{M}_{j,1}$ as above is called a *stable completion* of $\mathbf{M}_{j,0}$.

It is clear from the discussion in Section 3.3 that not any arbitrary complement basis Ψ_j of Φ_j is suitable. Thus $\mathbf{M}_{j,1}$ has to be chosen judiciously. It is important to note that

the stability of $\Phi_j \cup \Psi_j$ as a basis for $S(\Phi_{j+1})$ is by no means sufficient to ensure that $\Psi = \bigcup_{j=-1}^{\infty} \Psi_j$ is a Riesz basis for H.

The point of view taken in the sequel is as follows. In many cases *some* stable completion $\hat{\mathbf{M}}_{j,1}$ of $\mathbf{M}_{j,0}$ is available along with the inverse $\hat{\mathbf{G}}_j$ which may, however, induce a complement basis $\hat{\Psi}_j$ which is not yet satisfactory. The idea is to generate from such an *initial* completion $\hat{\mathbf{M}}_{j,1}$ another stable completion $\mathbf{M}_{j,1}$ which possesses better properties [CDP, Sw].

As a first step one has to parametrize the set of *all* stable completions in order to see then how to choose the parameters for a suitable one.

The assumed stability allows us to treat all operators like finite matrices which, of course, is the case in practical situations although the setting covers, in principle, infinite collections Φ_j, Ψ_j. The key observation is that whenever $\mathbf{L}_j \in [\ell_2(\nabla_j), \ell_2(\Delta_j)]$, $\mathbf{K}_j, \mathbf{K}_j^{-1} \in [\ell_2(\nabla_j), \ell_2(\nabla_j)]$ are uniformly bounded, then so are

$$\begin{pmatrix} \mathbf{I} & \mathbf{L}_j \\ 0 & \mathbf{K}_j \end{pmatrix}, \begin{pmatrix} \mathbf{I} & -\mathbf{L}_j\mathbf{K}_j^{-1} \\ 0 & \mathbf{K}_j^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{L}_j \\ 0 & \mathbf{K}_j \end{pmatrix}^{-1} \tag{5.43}$$

so that

$$\mathbf{M}_j = \hat{\mathbf{M}}_j \begin{pmatrix} \mathbf{I} & \mathbf{L}_j \\ 0 & \mathbf{K}_j \end{pmatrix}, \quad \mathbf{G}_j = \begin{pmatrix} \mathbf{I} & -\mathbf{L}_j\mathbf{K}_j^{-1} \\ 0 & \mathbf{K}_j^{-1} \end{pmatrix} \hat{\mathbf{G}}_j \tag{5.44}$$

still satisfy $\mathbf{M}_j\mathbf{G}_j = \mathbf{I}$ and

$$\mathbf{M}_{j,1} = \mathbf{M}_{j,0}\mathbf{L}_j + \hat{\mathbf{M}}_{j,1}\mathbf{K}_j \tag{5.45}$$

is another stable completion of $\mathbf{M}_{j,0}$. By (5.44) the blocks of $\mathbf{G}_j$ are given by

$$\mathbf{G}_{j,0} = \hat{\mathbf{G}}_{j,0} - \mathbf{L}_j\mathbf{K}_j^{-1}\hat{\mathbf{G}}_{j,1}, \quad \mathbf{G}_{j,1} = \mathbf{K}_j^{-1}\hat{\mathbf{G}}_{j,1}. \tag{5.46}$$

Moreover, one can show that whenever $\mathbf{M}_{j,1}$ and $\hat{\mathbf{M}}_{j,1}$ are any two stable completions of $\mathbf{M}_{j,0}$ then they are related by (5.44) for some $\mathbf{L}_j$, $\mathbf{K}_j$ with the properties stated above [CDP].

This can be used in several ways. The point of view taken in [Sw] may be roughly described as follows. Given some 'simple' initial bases Φ_j with refinement matrices $\mathbf{M}_{j,0}$, e.g. Haar type functions, as well as some stable completions $\hat{\mathbf{M}}_{j,1}$, choose *only* some $\mathbf{L}_j$ to form another stable completion of the form (5.45)

$$\mathbf{M}_{j,1} = \mathbf{M}_{j,0}\mathbf{L}_j + \hat{\mathbf{M}}_{j,1}$$

which means that the new complement functions $\psi_{j,k}$ take the form

$$\psi_{j,k} = \sum_{m \in \Delta_j} \ell_{m,k}\varphi_{j,m} + \hat{\psi}_{j,k},$$

i.e., they are obtained by adding a linear combination of coarse grid functions to the previous complement basis functions. This can be exploited to increase the efficiency of corresponding multiscale transformations $\mathbf{T}_j$ significantly [Sw]. Next one can exchange the roles of $\mathbf{M}_j$ and $\mathbf{G}_j^*$ (the formal adjoint of $\mathbf{G}_j$) which results in modifying the original

bases Φ_j. Possibly alternating this procedure one hopes to end up with biorthogonal systems with more desirable properties pertaining to regularity or vanishing moments.

A somewhat different point of view which will be taken up here is to keep the spaces $S_j = S(\Phi_j)$ fixed and generate from some initial completion $\hat{\mathbf{M}}_{j,1}$ of $\mathbf{M}_{j,0}$ in *one step* a certain 'target' completion $\mathbf{M}_{j,1}$. From the discussion in Section 3.3 it is clear that an appropriate complement basis $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$ must fulfill

$$(Q_{j+1} - Q_j)\psi_{j,k} = \psi_{j,k}, \quad k \in \nabla_j, \tag{5.47}$$

where the $Q_j : H \to S_j$ are linear uniformly H-bounded projectors satisfying

$$Q_\ell Q_j = Q_\ell, \quad \ell \leq j. \tag{5.48}$$

It is not hard to verify that in terms of the generating basis Φ_j such Q_j must have the form

$$Q_j v = \sum_{k \in \Delta_j} \langle v, \tilde{\varphi}_{j,k} \rangle \, \varphi_{j,k} \tag{5.49}$$

where $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k} : k \in \Delta_j\}$ is biorthogonal to Φ_j, i.e.,

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,k'} \rangle = \delta_{k,k'}, \quad k, k' \in \Delta_j, \tag{5.50}$$

and refinable

$$\tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}. \tag{5.51}$$

Note that (5.51) means that the ranges $\tilde{S}_j = S(\tilde{\Phi}_j)$ of the adjoints Q_j^* of Q_j are *also* nested. Once the projectors Q_j satisfying (5.48) are known one has to identify complement basis Ψ_j, $\tilde{\Psi}_j$ such that

$$\left\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \right\rangle = \delta_{(j,k),(j',k')}, \quad (j,k),(j',k') \in \nabla = \bigcup_{j=-1}^{\infty} (\{j\} \times \nabla_j). \tag{5.52}$$

Given some initial stable completion $\hat{\mathbf{M}}_{j,1}$ of $\mathbf{M}_{j,0}$ this can indeed be facilitated as follows [CDP].

Theorem 5.2 *Let* $\{\Phi_j\}$, $\{\tilde{\Phi}_j\}$, $\mathbf{M}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$ *be related by* (5.38), (5.50) *and* (5.51). *Suppose that* $\hat{\mathbf{M}}_{j,1}$ *is some stable completion of* $\mathbf{M}_{j,0}$ *and that* $\hat{\mathbf{G}}_j = \hat{\mathbf{M}}_j^{-1}$. *Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^*\right) \hat{\mathbf{M}}_{j,1} \tag{5.53}$$

is a stable completion of $\mathbf{M}_{j,0}$ *and* $\mathbf{G}_j = \mathbf{M}_j^{-1}$ *has the form*

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^* \\ \hat{\mathbf{G}}_{j,1} \end{pmatrix}.$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j = \hat{\mathbf{G}}_{j,1} \tilde{\Phi}_{j+1}$$

form biorthogonal systems, i.e.,

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \mathbf{0}, \tag{5.54}$$

where $\langle \Psi_j, \tilde{\Psi}_j \rangle := (\langle \psi_{j,k}, \tilde{\psi}_{j,k'} \rangle)_{k,k' \in \nabla_j}$, *so that, in particular,*

$$\left\langle \Psi_j, \tilde{\Psi}_{j'} \right\rangle = \delta_{j,j'} \mathbf{I}, \quad j, j' \in I\!N_0. \tag{5.55}$$

Some applications of this result will be outlined below.

There is yet another useful observation in the same spirit concerning changes of bases within each space S_j. Suppose $\Phi_j^{(0)}$ is the 'old' basis for S_j with refinement matrix $\mathbf{M}_{j,0}^0$, stable completion $\mathbf{M}_{j,1}^{(0)}$ and inverse $\mathbf{G}_j^{(0)}$.

Remark 5.1 [DS1] *Given a 'new' basis*

$$\Phi_j^{(n)} = \mathbf{C}_j \Phi_j^{(0)} \tag{5.56}$$

of S_j, *the corresponding refinement matrix* $\mathbf{M}_{j,0}^{(n)}$ *and stable completion* $\mathbf{M}_{j,1}^{(n)}$ *are given by*

$$\begin{aligned} \mathbf{M}_{j,0}^{(n)} &= \mathbf{C}_{j+1}^{-T} \mathbf{M}_{j,0}^{(0)} \mathbf{C}_j^T \\ \mathbf{M}_{j,1}^{(n)} &= \mathbf{C}_{j+1}^{-T} \mathbf{M}_{j,1}^{(0)} \end{aligned} \tag{5.57}$$

and

$$\left(\mathbf{M}_j^{(n)} \right)^{-1} = \begin{pmatrix} \mathbf{C}_j^{-T} \mathbf{G}_{j,0}^{(0)} \mathbf{C}_{j+1}^T \\ \mathbf{G}_{j,1}^{(0)} \mathbf{C}_{j+1}^T \end{pmatrix} =: \mathbf{G}_j^{(n)}. \tag{5.58}$$

5.6 Wavelets on Surfaces in $I\!R^3$

We briefly indicate next how the above concepts can be employed in the context of problem (III), namely, boundary integral equations on Lipschitz manifolds Γ in $I\!R^3$. We refer to the setting described in Section 4.2, i.e., Γ has the form (4.9) where each Γ_i is the parametric image of the unit square $\square := [0,1] \times [0,1]$ in $I\!R^2$. To construct a multiresolution sequence $\mathcal{S}$ for $L_2(\Gamma)$ with the properties worked out in Sections 4.1, 4.2, 4.3 one may proceed as follows:

1. Construct a biorthogonal multiresolution for $L_2([0,1])$ where the basis functions satisfy certain boundary conditions.
2. Tensor products yield biorthogonal multiresolution sequences on $\square$.
3. With the aid of the mappings $\kappa_i : \square \to \Gamma_i$ these can be lifted to Γ where the boundary conditions mentioned in 1. allow us to glue these local multiresolution sequence together.

First note that in essentially all cases of interest covering the single and double layer potential operator as well as the hypersingular operator it suffices to have that Φ_j and $\tilde{\Phi}_j$ are continuous. Discontinuous bases such as Haar type bases or multiwavelets have a more restricted applicability and the disadvantage that the lack of global smoothness causes a relatively strong increase of $\dim S_j$ without raising the degree of accuracy. What matters is the degree of vanishing moments (see (4.10)) controlling the compression rate. This suggests the following approach. Let

$$\theta(x) = \begin{cases} 1+x, & x \in [-1,0], \\ 1-x, & x \in [0,1], \\ 0, & \text{else}, \end{cases}$$

the piecewise linear hat function. It is known from [CDF] that for any $d^* \in \mathbb{N}$, d^* even, there exists a refinable even function $\tilde{\theta} = \tilde{\theta}_{d^*} \in L_2(\mathbb{R})$ with support in $[-d^*, d^*]$ such that

$$\left\langle \theta, \tilde{\theta}(\cdot - k) \right\rangle_{\mathbb{R}} = \delta_{0,k}, \quad k \in \mathbb{Z}, \tag{5.59}$$

where

$$\langle f, g \rangle_A := \int_A f(x)\overline{g(x)}\,\mathrm{d}x$$

and $\tilde{\theta}$ is exact of degree $d^* - 1$ (see (5.7)).

By the strategy outlined in Section 5.2 one can establish the following specific facts [DS1].

Theorem 5.3 *Let for θ, $\tilde{\theta}$ as above and fixed even d^**

$$K_{j,L} = \{0, \dots, d^* - 1\}, \quad K_{j,R} = \{2^j - d^* + 1, \dots, 2^j\}, \quad K_{j,I} = \{d^*, \dots, 2^j - d^*\}$$

and

$$K_j = K_{j,L} \cup K_{j,I} \cup K_{j,R}.$$

Moreover, let

$$\hat{K}_{j,L} = \{-d^* + 1, \dots, d^* - 1\}, \quad \hat{K}_{j,R} = \{2^j - d^* + 1, \dots, 2^j + d^* - 1\}.$$

Then there exist coefficients $g^V_{k,\ell}$, $\ell, k \in K_{j,V}$, $\tilde{g}^V_{k,\ell}$, $\ell \in \hat{K}_{j,V}$, $k \in K_{j,V}$, $V \in \{L, R\}$, independent of j with $2^j > 3d^* - 2$, such that the functions

$$\theta_{j,k} := \sum_{\ell \in K_{j,V}} g^V_{k,\ell}\, 2^{j/2}\theta(2^j \cdot - \ell)|_{[0,1]}, \quad \tilde{\theta}_{j,k} := \sum_{\ell \in \hat{K}_{j,V}} \tilde{g}^V_{k,\ell}\, 2^{j/2}\tilde{\theta}(2^j \cdot - \ell)|_{[0,1]},$$

$k \in K_{j,V}$, $V \in \{L, R\}$, and

$$\theta_{j,k} := 2^{j/2}\theta(2^j \cdot - k), \quad \tilde{\theta}_{j,k} := 2^{j/2}\tilde{\theta}(2^j \cdot - k), \quad k \in K_{j,I},$$

have the following properties:

(i) The collections $\Theta_j = \{\theta_{j,k} : k \in K_j\}$, $\tilde{\Theta}_j = \{\tilde{\theta}_{j,k} : k \in K_j\}$ are biorthogonal, i.e.,

$$\left\langle \theta_{j,k}, \tilde{\theta}_{j,k'} \right\rangle_{[0,1]} = \delta_{k,k'}, \quad k, k' \in K_j.$$

(ii) One has

$$\theta_{j,k}(0) = \theta_{j,k}(1) = \tilde{\theta}_{j,k}(0) = \tilde{\theta}_{j,k}(1) = 0, \quad k \in K_j \setminus \{0, 2^j\}. \tag{5.60}$$

(iii) The symmetry relations

$$\theta_{j,k}(x) = \theta_{j,2^j-k}(1-x), \quad \tilde{\theta}_{j,k}(x) = \tilde{\theta}_{j,2^j-k}(1-x), \quad k = 0, \ldots, d^* - 1, \tag{5.61}$$

hold.

(iv) The spaces $S(\Theta_j)$, $S(\tilde{\Theta}_j)$ are nested, i.e., the Θ_j and $\tilde{\Theta}_j$ are refinable.

(v) One has for $2^j > 3d^* - 2$

$$\Pi_2([0,1]) \subset S(\Theta_j), \quad \Pi_{d^*}([0,1]) \subset S(\tilde{\Theta}_j).$$

We refer to [DS1] for the concrete determination of the coefficients $g^V_{k,\ell}$, $\tilde{g}^V_{k,\ell}$ above.

At this point one is exactly in the situation described in the previous section. In fact, the biorthogonal systems Θ_j, $\tilde{\Theta}_j$ induce projectors of the form (5.49) which satisfy (5.48). To construct now the corresponding biorthogonal wavelets one can employ the concept of stable completions. In fact, for the hat function basis

$$\Theta_j^{(0)} = \left\{ 2^{j/2}\theta(2^j \cdot -k)|_{[0,1]} \,:\, k = 0, \ldots, 2^j \right\}$$

a simple stable completion corresponds to the hierarchical complement bases

$$\left\{ 2^{j/2}\theta(2^{j+1} \cdot -k) \,:\, k = 2\ell + 1,\, \ell = 0, \ldots 2^j - 1 \right\}.$$

Remark 5.1 yields then a stable completion relative to the bases $\Theta_j^{(n)} = \Theta_j$. One is now in the position to apply Theorem 5.2 providing the desired biorthogonal wavelet bases on $[0,1]$. Forming tensor products of these univariate wavelets and transporting these functions on Γ via the parametric mappings κ_i can be shown to give rise to a multiscale bases $\Psi = \{\psi_{j,k} \,:\, k \in \nabla\}$ on Γ with the following properties [DS1]:

1. The elements of Φ_j and $\tilde{\Phi}_j$ are continuous. This is a consequence of (5.60).
2. The Ψ have patchwise vanishing moments of order d^* in the sense of (4.10). This follows from Theorem 5.3(v).
3. The generator bases Φ_j, $\tilde{\Phi}_j$ obtained by lifting and gluing are biorthogonal relative to the inner product

$$(f, g) := \sum_{i=1}^{N} \int_{\square} (f \circ \kappa_i)(y)\, (g \circ \kappa_i)(y)\, \mathrm{d}y.$$

4. Defining for $s \geq 0$ the scale of norms

$$\|v\|_s := \left(\sum_{i=1}^{N} \|v\|^2_{H^s(\Omega)} \right)^{1/2}$$

and denoting by $\|\cdot\|_{-s}$, $s > 0$, the respective dual norms, the corresponding projectors Q_j defined by (5.49) give rise to the norm equivalences

$$\|v\|_s \sim \left(\sum_{j=j_0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|^2_{L_2(\Gamma)} \right)^{1/2}, \quad s \in (-1, 1),$$

where $Q_{j_0-1} := 0$.

6 Adaptivity and Multiscale Bases

The discussion has been confined so far to the efficient solution of systems corresponding to an a-priori fixed sequence of trial spaces which are typically related to uniform mesh refinements. A further reduction of the solution complexity may be expected by *adapting* the trial spaces to the particular problem at hand. Adaptive procedures have been studied on various levels of generality. The messages conveyed by some of these results differ significantly as to the actual success of such techniques [DY, KN1, KN2, TWW] on a principal level. On the other hand, practical experience gained in connection with solving partial differential equations clearly confirms a tremendeous power of adaptive techniques which make certain problems computable at all [HJ]. Adaptive refinement strategies based on *a-posteriori local error estimators* or *indicators* have been particularly far developed in a finite element context [BEK, BR, HJ, Ve]. Nevertheless, rigorous proofs for the precise convergence behavior of such techniques are by far less advanced. These known a-posteriori error indicators depend strongly on the particular choice of trial space as well as on the spatial dimension and therefore do not carry over into the present multiscale basis oriented setting. On the other hand, most of the techniques employed in connection with wavelet analysis (so far mainly for univariate problems) assume essential knowledge about the singularity [LPT, BMR, JMP] and are therefore not adaptive in the strict sense.

In the following we will briefly review some recent first results about adaptive strategies for multiscale basis methods which rely on an a-posteriori error estimate and can be proved to converge for a wide range of elliptic problems [DDHS].

We adhere to the general setting from Sections 3 and 4.1. A key role will again be played by norm equivalences of the form (3.18) or (3.25). Thus, we will assume again that Ψ and $\tilde{\Psi}$ are stable biorthogonal Riesz bases giving rise to such norm equivalences. Instead of seeking for an approximate solution of (1.2) in a *full* space S_j according to (4.2), the goal is to break S_j into small subspaces that capture possibly much information about the particular solution. To describe this, it is convenient to employ the following notation. As before, let $\nabla = \bigcup_{j=-1}^{\infty}(\{j\} \times \nabla_j)$. By Λ we will denote finite subsets of ∇, i.e., each $\lambda \in \Lambda$ has the form $\lambda = (j, k)$ where $|\lambda| := j$ indicates the refinement level. Accordingly, we set $S_\Lambda = S(\{\psi_\lambda : \lambda \in \Lambda\})$ and

$$Q_\Lambda v := \sum_{\lambda \in \Lambda} \left\langle v, \tilde{\psi}_\lambda \right\rangle \psi_\lambda$$

denotes the corresponding projector. Let u_Λ denote the solution of

$$Q^*_\Lambda A u_\Lambda = Q^*_\Lambda f \tag{6.1}$$

where we assume solvability of (6.1) for any $\Lambda \subset \nabla$ which is e.g. the case under the assumption (2.1). In fact, we will assume that for $\tau = r/2$

$$\|Av\|_{H^{-\tau}} \sim \|v\|_{H^\tau}, \quad \|Q_\Lambda^* A Q_\Lambda v\|_{H^{-\tau}} \sim \|Q_\Lambda v\|_{H^\tau}, \quad v \in H^\tau. \tag{6.2}$$

We wish to base the adaptive selection of sets $\Lambda \subset \nabla$ on estimating the error in the 'energy' norm $\|u - u_\Lambda\|_{H^\tau}$ which by (6.2) satisfies

$$\|r_\Lambda\|_{H^{-\tau}} \sim \|u - u_\Lambda\|_{H^\tau} \tag{6.3}$$

where

$$r_\Lambda := A(u - u_\Lambda) = f - Au_\Lambda$$

is the residual. When $|\tau| < \{\gamma, \tilde{\gamma}\}$ (4.5) and (3.25) tell us that $\|r_\Lambda\|_{H^{-\tau}}$ can, in principle, be evaluated via

$$\|r_\Lambda\|_{H^{-\tau}} \sim \left(\sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2\tau|\lambda|} |\langle r_\Lambda, \psi_\lambda \rangle|^2 \right)^{1/2} =: \left(\sum_{\lambda \in \nabla \setminus \Lambda} \delta_\lambda^2 \right)^{1/2}. \tag{6.4}$$

Writing

$$u_\Lambda = \sum_{\lambda' \in \Lambda} u_{\lambda'} \psi_{\lambda'}, \quad f = \sum_{\lambda \in \nabla} f_\lambda \tilde{\psi}_\lambda, \quad f_\lambda = \langle f, \psi_\lambda \rangle,$$

one has

$$\delta_\lambda = 2^{-|\lambda|\tau} \left| f_\lambda - \sum_{\lambda' \in \Lambda} \langle A\psi_{\lambda'}, \psi_\lambda \rangle \, u_\lambda \right|. \tag{6.5}$$

Employing arguments similar to those used in connection with matrix compression, one can show [DDHS] that one can replace for any given $\varepsilon > 0$ the infinite sum on the right hand side of (6.4) by an expression $\left(\sum_{\lambda \in \nabla \setminus \Lambda} d_\lambda^2 \right)^{1/2}$ such that

$$\|u - u_\Lambda\|_{H^\tau} \lesssim \left(\sum_{\lambda \in \nabla \setminus \Lambda} d_\lambda^2 \right)^{1/2} + \varepsilon \|f\|_{H^{-\tau}} \tag{6.6}$$

and

$$\left(\sum_{\lambda \in \nabla \setminus \Lambda} d_\lambda^2 \right)^{1/2} \lesssim \|u - u_\Lambda\|_{H^\tau} + \varepsilon \|f\|_{H^{-\tau}}. \tag{6.7}$$

Here

$$d_\lambda = d_\lambda(\Lambda, \epsilon) := 2^{-|\lambda|\tau} \left| f_\lambda - \sum_{\lambda' \in \Lambda \cap \mathcal{N}_{\lambda,\epsilon}} \langle A\psi_{\lambda'}, \psi_\lambda \rangle \, u_\lambda \right|$$

and $\mathcal{N}_{\lambda,\epsilon}$ is a "neighborhood" of λ which is determined by the decay behavior of the entries $\langle A\psi_{\lambda'}, \psi_\lambda \rangle$ estimated e.g. by (4.12) as well as by a given tolerance $\varepsilon > 0$ [DDHS]. Specifically, denoting by

$$e_\lambda := \sum_{\lambda' \in \Lambda \setminus \mathcal{N}_{\lambda,\epsilon}} \langle A\psi_{\lambda'}, \psi_\lambda \rangle \, u_{\lambda'}$$

the portion of δ_λ which is neglected when passing to the d_λ, it can be shown that

$$\left(\sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2\tau|\lambda|} \, |e_\lambda|^2 \right)^{1/2} \leq \beta \varepsilon \, \|f\|_{H^{-\tau}} \, . \tag{6.8}$$

Estimates of this type have been presented first in [**?**] for the special case of a second order two point boundary value problem.

Since $\Lambda \cap \mathcal{N}_{\lambda,\epsilon} \neq \emptyset$ for only finitely many $\lambda \in \nabla \setminus \Lambda$ the d_λ are determined for most $\lambda \in \nabla \setminus \Lambda$ by properties of f. This suggests considering the quantities

$$a_\lambda = a_\lambda(\Lambda, \epsilon) := 2^{-\tau|\lambda|} \left| \sum_{\lambda' \in \Lambda \cap \mathcal{N}_{\lambda,\epsilon}} \langle A\psi_{\lambda'}, \psi_\lambda \rangle \, u_\lambda \right| .$$

The idea is now to choose $\tilde{\Lambda} \supset \Lambda$ such that the a_λ, $\lambda \in \hat{\Lambda} \setminus \Lambda$, represent most of the error. For simplicity we will assume that A is positive definite and self-adjoint. Thus $\|v\| := \langle Av, v \rangle^{1/2} \sim \|v\|_{H^\tau}$. The key result from [DDHS] may then be formulated as follows.

Theorem 6.1 *Given* $\mathrm{eps} > 0$ *there exists a* $\mu > 0$ *such that for any finite set* $\Lambda \subset \nabla$ *with*

$$\|Q_\Lambda^* f - f\|_{H^{-\tau}} \sim \left(\sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2(\tau|\lambda|-1)} \, |f_\lambda|^2 \right)^{1/2} < \mu \, \mathrm{eps}$$

and

$$\varepsilon := \frac{\mu \cdot \mathrm{eps}}{\beta \, \|f\|_{H^{-\tau}}},$$

any finite index set $\tilde{\Lambda} \subset \nabla$, $\Lambda \subset \tilde{\Lambda}$, *satisfying*

$$\left(\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} \geq \frac{1}{2} \left(\sum_{\lambda \in \nabla \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} ,$$

ensures that for some $\kappa \in (0,1)$ *either*

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \, \|u - u_\Lambda\|$$

or

$$\left(\sum_{\lambda \in \nabla \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} < \mathrm{eps} \, .$$

For a concrete adaptive scheme which is guaranteed to converge under the above circumstances, the reader is referred to [DDHS].

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